6 POLYNOMIALS

Figure 6.1 Architects use polynomials to design curved shapes such as this suspension bridge, the Silver Jubilee bridge in Halton, England.

Chapter Outline

Introduction

We have seen that the graphs of linear equations are straight lines. Graphs of other types of equations, called polynomial equations, are curves, like the outline of this suspension bridge. Architects use polynomials to design the shape of a bridge like this and to draw the blueprints for it. Engineers use polynomials to calculate the stress on the bridge's supports to ensure they are strong enough for the intended load. In this chapter, you will explore operations with and properties of polynomials.

6.1 Add and Subtract Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Identify polynomials, monomials, binomials, and trinomials
- Determine the degree of polynomials
- Add and subtract monomials
- Add and subtract polynomials
- Evaluate a polynomial for a given value

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $8x + 3x$.
   If you missed this problem, review **Example 1.24**.
2. Subtract: $(5n + 8) - (2n - 1)$.
   If you missed this problem, review **Example 1.139**.

3. Write in expanded form: a^5.
If you missed this problem, review **Example 1.14**.

Identify Polynomials, Monomials, Binomials and Trinomials

You have learned that a *term* is a constant or the product of a constant and one or more variables. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a monomial. Some examples of monomial are $8,\ -2x^2,\ 4y^3,\ \text{and } 11z^7$.

Monomials

A **monomial** is a term of the form ax^m, where a is a constant and m is a positive whole number.

A monomial, or two or more monomials combined by addition or subtraction, is a polynomial. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term. A binomial has exactly two terms, and a trinomial has exactly three terms. There are no special names for polynomials with more than three terms.

Polynomials

polynomial—A monomial, or two or more monomials combined by addition or subtraction, is a polynomial.

- **monomial**—A polynomial with exactly one term is called a monomial.
- **binomial**—A polynomial with exactly two terms is called a binomial.
- **trinomial**—A polynomial with exactly three terms is called a trinomial.

Here are some examples of polynomials.

Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	-13
Binomial	$a+7$	$4b-5$	y^2-16	$3x^3-9x^2$
Trinomial	$x^2-7x+12$	$9y^2+2y-8$	$6m^4-m^3+8m$	z^4+3z^2-1

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the "family" of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special polynomials and just call all the rest *polynomials*.

EXAMPLE 6.1

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial.

ⓐ $4y^2-8y-6$ ⓑ $-5a^4b^2$ ⓒ $2x^5-5x^3-9x^2+3x+4$ ⓓ $13-5m^3$ ⓔ q

Solution

	Polynomial	Number of terms	Type
(a)	$4y^2-8y-6$	3	Trinomial
(b)	$-5a^4b^2$	1	Monomial
(c)	$2x^5-5x^3-9x^2+3x+4$	5	Polynomial
(d)	$13-5m^3$	2	Binomial
(e)	q	1	Monomial

TRY IT : : 6.1 Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

ⓐ $5b$ ⓑ $8y^3-7y^2-y-3$ ⓒ $-3x^2-5x+9$ ⓓ $81-4a^2$ ⓔ $-5x^6$

TRY IT : : 6.2 Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

ⓐ $27z^3 - 8$ ⓑ $12m^3 - 5m^2 - 2m$ ⓒ $\frac{5}{6}$ ⓓ $8x^4 - 7x^2 - 6x - 5$ ⓔ $-n^4$

Determine the Degree of Polynomials

The degree of a polynomial and the degree of its terms are determined by the exponents of the variable.

A monomial that has no variable, just a constant, is a special case. The degree of a constant is 0—it has no variable.

Degree of a Polynomial

The **degree of a term** is the sum of the exponents of its variables.

The **degree of a constant** is 0.

The **degree of a polynomial** is the highest degree of all its terms.

Let's see how this works by looking at several polynomials. We'll take it step by step, starting with monomials, and then progressing to polynomials with more terms.

Monomial	14	$8y^2$	$-9x^3y^5$	$-13a$
Degree	0	2	8	1
Binomial	$a+7$	$4b^2-5b$	x^2y^2-16	$3n^3-9n^2$
Degree of each term	0 1	2 1	4 0	3 2
Degree of polynomial	1	2	4	3
Trinomial	$x^2-7x+12$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of each term	2 1 0	2 2 2	4 5 6	4 2 0
Degree of polynomial	2	2	6	4
Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	1 0	2 1 0	4 3 2 1 0	
Degree of polynomial	1	2	4	

A polynomial is in **standard form** when the terms of a polynomial are written in descending order of degrees. Get in the habit of writing the term with the highest degree first.

EXAMPLE 6.2

Find the degree of the following polynomials.

ⓐ $10y$ ⓑ $4x^3 - 7x + 5$ ⓒ -15 ⓓ $-8b^2 + 9b - 2$ ⓔ $8xy^2 + 2y$

Solution

ⓐ

	$10y$
The exponent of y is one. $y = y^1$	The degree is 1.

ⓑ

	$4x^3 - 7x + 5$
The highest degree of all the terms is 3.	The degree is 3.

ⓒ

	-15
The degree of a constant is 0.	The degree is 0.

ⓓ

	$-8b^2 + 9b - 2$
The highest degree of all the terms is 2.	The degree is 2.

ⓔ

	$8xy^2 + 2y$
The highest degree of all the terms is 3.	The degree is 3.

TRY IT : : 6.3 Find the degree of the following polynomials:

ⓐ $-15b$ ⓑ $10z^4 + 4z^2 - 5$ ⓒ $12c^5 d^4 + 9c^3 d^9 - 7$ ⓓ $3x^2 y - 4x$ ⓔ -9

TRY IT : : 6.4 Find the degree of the following polynomials:

ⓐ 52 ⓑ $a^4 b - 17a^4$ ⓒ $5x + 6y + 2z$ ⓓ $3x^2 - 5x + 7$ ⓔ $-a^3$

Add and Subtract Monomials

You have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponent. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficient.

EXAMPLE 6.3

Add: $25y^2 + 15y^2$.

Solution

	$25y^2 + 15y^2$
Combine like terms.	$40y^2$

TRY IT : : 6.5 Add: $12q^2 + 9q^2$.

TRY IT : : 6.6 Add: $-15c^2 + 8c^2$.

EXAMPLE 6.4

Subtract: $16p - (-7p)$.

Solution

	$16p - (-7p)$
Combine like terms.	$23p$

TRY IT : : 6.7 Subtract: $8m - (-5m)$.

TRY IT : : 6.8 Subtract: $-15z^3 - \left(-5z^3\right)$.

Remember that like terms must have the same variables with the same exponents.

EXAMPLE 6.5

Simplify: $c^2 + 7d^2 - 6c^2$.

Solution

	$c^2 + 7d^2 - 6c^2$
Combine like terms.	$-5c^2 + 7d^2$

TRY IT : : 6.9 Add: $8y^2 + 3z^2 - 3y^2$.

TRY IT : : 6.10 Add: $3m^2 + n^2 - 7m^2$.

EXAMPLE 6.6

Simplify: $u^2 v + 5u^2 - 3v^2$.

Solution

	$u^2 v + 5u^2 - 3v^2$
There are no like terms to combine.	$u^2 v + 5u^2 - 3v^2$

TRY IT : : 6.11 Simplify: $m^2 n^2 - 8m^2 + 4n^2$.

TRY IT : : 6.12 Simplify: $pq^2 - 6p - 5q^2$.

Add and Subtract Polynomials

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

EXAMPLE 6.7

Find the sum: $\left(5y^2 - 3y + 15\right) + \left(3y^2 - 4y - 11\right)$.

Solution

Identify like terms.	$(\underline{\underline{5y^2}} - \underline{3y} + 15) + (\underline{\underline{3y^2}} - \underline{4y} - 11)$
Rearrange to get the like terms together.	$\underline{\underline{5y^2 + 3y^2}} - \underline{3y - 4y} + 15 - 11$
Combine like terms.	$8y^2 - 7y + 4$

TRY IT : : 6.13 Find the sum: $\left(7x^2 - 4x + 5\right) + \left(x^2 - 7x + 3\right)$.

TRY IT : : 6.14 Find the sum: $\left(14y^2 + 6y - 4\right) + \left(3y^2 + 8y + 5\right)$.

EXAMPLE 6.8

Find the difference: $(9w^2 - 7w + 5) - (2w^2 - 4)$.

Solution

	$(9w^2 - 7w + 5) - (2w^2 - 4)$
Distribute and identify like terms.	$\underline{\underline{9w^2}} - \underline{7w} + \underset{\sim}{5} - \underline{\underline{2w^2}} + \underset{\sim}{4}$
Rearrange the terms.	$\underline{\underline{9w^2 - 2w^2}} - \underline{7w} + \underset{\sim}{5} + \underset{\sim}{4}$
Combine like terms.	$7w^2 - 7w + 9$

TRY IT : : 6.15 Find the difference: $(8x^2 + 3x - 19) - (7x^2 - 14)$.

TRY IT : : 6.16 Find the difference: $(9b^2 - 5b - 4) - (3b^2 - 5b - 7)$.

EXAMPLE 6.9

Subtract: $(c^2 - 4c + 7)$ from $(7c^2 - 5c + 3)$.

Solution

	Subtract $(c^2 - 4c + 7)$ from $(7c^2 - 5c + 3)$.
	$(7c^2 - 5c + 3) - (c^2 - 4c + 7)$
Distribute and identify like terms.	$\underline{\underline{7c^2}} - \underline{5c} + \underset{\sim}{3} - \underline{\underline{c^2}} + \underline{4c} - \underset{\sim}{7}$
Rearrange the terms.	$\underline{\underline{7c^2 - c^2}} - \underline{5c + 4c} + \underset{\sim}{3} - \underset{\sim}{7}$
Combine like terms.	$6c^2 - c - 4$

TRY IT : : 6.17 Subtract: $(5z^2 - 6z - 2)$ from $(7z^2 + 6z - 4)$.

TRY IT : : 6.18 Subtract: $(x^2 - 5x - 8)$ from $(6x^2 + 9x - 1)$.

EXAMPLE 6.10

Find the sum: $(u^2 - 6uv + 5v^2) + (3u^2 + 2uv)$.

Solution

	$(u^2 - 6uv + 5v^2) + (3u^2 + 2uv)$
Distribute.	$u^2 - 6uv + 5v^2 + 3u^2 + 2uv$
Rearrange the terms, to put like terms together.	$u^2 + 3u^2 - 6uv + 2uv + 5v^2$
Combine like terms.	$4u^2 - 4uv + 5v^2$

TRY IT : : 6.19 Find the sum: $\left(3x^2 - 4xy + 5y^2\right) + \left(2x^2 - xy\right)$.

TRY IT : : 6.20 Find the sum: $\left(2x^2 - 3xy - 2y^2\right) + \left(5x^2 - 3xy\right)$.

EXAMPLE 6.11

Find the difference: $\left(p^2 + q^2\right) - \left(p^2 + 10pq - 2q^2\right)$.

Solution

	$\left(p^2 + q^2\right) - \left(p^2 + 10pq - 2q^2\right)$
Distribute.	$p^2 + q^2 - p^2 - 10pq + 2q^2$
Rearrange the terms, to put like terms together.	$p^2 - p^2 - 10pq + q^2 + 2q^2$
Combine like terms.	$-10pq^2 + 3q^2$

TRY IT : : 6.21 Find the difference: $\left(a^2 + b^2\right) - \left(a^2 + 5ab - 6b^2\right)$.

TRY IT : : 6.22 Find the difference: $\left(m^2 + n^2\right) - \left(m^2 - 7mn - 3n^2\right)$.

EXAMPLE 6.12

Simplify: $\left(a^3 - a^2 b\right) - \left(ab^2 + b^3\right) + \left(a^2 b + ab^2\right)$.

Solution

	$\left(a^3 - a^2 b\right) - \left(ab^2 + b^3\right) + \left(a^2 b + ab^2\right)$
Distribute.	$a^3 - a^2 b - ab^2 - b^3 + a^2 b + ab^2$
Rearrange the terms, to put like terms together.	$a^3 - a^2 b + a^2 b - ab^2 + ab^2 - b^3$
Combine like terms.	$a^3 - b^3$

TRY IT : : 6.23 Simplify: $\left(x^3 - x^2 y\right) - \left(xy^2 + y^3\right) + \left(x^2 y + xy^2\right)$.

TRY IT : : 6.24 Simplify: $\left(p^3 - p^2 q\right) + \left(pq^2 + q^3\right) - \left(p^2 q + pq^2\right)$.

Evaluate a Polynomial for a Given Value

We have already learned how to evaluate expressions. Since polynomials are expressions, we'll follow the same procedures to evaluate a polynomial. We will substitute the given value for the variable and then simplify using the order of operations.

EXAMPLE 6.13

Evaluate $5x^2 - 8x + 4$ when

ⓐ $x = 4$ ⓑ $x = -2$ ⓒ $x = 0$

Solution

ⓐ $x = 4$

	$5x^2 - 8x + 4$
Substitute 4 for x.	$5(4)^2 - 8(4) + 4$
Simplify the exponents.	$5 \cdot 16 - 8(4) + 4$
Multiply.	$80 - 32 + 4$
Simplify.	52

ⓑ $x = -2$

	$5x^2 - 8x + 4$
Substitute -2 for x.	$5(-2)^2 - 8(-2) + 4$
Simplify the exponents.	$5 \cdot 4 - 8(-2) + 4$
Multiply.	$20 + 16 + 4$
Simplify.	40

ⓒ $x = 0$

	$5x^2 - 8x + 4$
Substitute 0 for x.	$5(0)^2 - 8(0) + 4$
Simplify the exponents.	$5 \cdot 0 - 8(0) + 4$
Multiply.	$0 + 0 + 4$
Simplify.	4

TRY IT : : 6.25 Evaluate: $3x^2 + 2x - 15$ when

ⓐ $x = 3$ ⓑ $x = -5$ ⓒ $x = 0$

TRY IT : : 6.26 Evaluate: $5z^2 - z - 4$ when

ⓐ $z = -2$ ⓑ $z = 0$ ⓒ $z = 2$

EXAMPLE 6.14

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250 foot tall building. Find the height after $t = 2$ seconds.

Solution

	$-16t^2 + 250$
Substitute $t = 2$.	$-16(2)^2 + 250$
Simplify.	$-16 \cdot 4 + 250$
Simplify.	$-64 + 250$
Simplify.	186
	After 2 seconds the height of the ball is 186 feet.

TRY IT : : 6.27

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 0$ seconds.

TRY IT : : 6.28

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 3$ seconds.

EXAMPLE 6.15

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side *x* feet and sides of height *y* feet. Find the cost of producing a box with $x = 4$ feet and $y = 6$ feet.

Solution

	$6x^2 + 15xy$
Substitute $x = 4$, $y = 6$.	$6(4)^2 + 15(4)(6)$
Simplify.	$6 \cdot 16 + 15(4)(6)$
Simplify.	$96 + 360$
Simplify.	456
	The cost of producing the box is $456.

TRY IT : : 6.29

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side *x* feet and sides of height *y* feet. Find the cost of producing a box with $x = 6$ feet and $y = 4$ feet.

TRY IT : : 6.30

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side *x* feet and sides of height *y* feet. Find the cost of producing a box with $x = 5$ feet and $y = 8$ feet.

MEDIA : :

Access these online resources for additional instruction and practice with adding and subtracting polynomials.

- **Add and Subtract Polynomials 1 (https://openstax.org/l/25Addsubtrpoly1)**
- **Add and Subtract Polynomials 2 (https://openstax.org/l/25Addsubtrpoly2)**
- **Add and Subtract Polynomial 3 (https://openstax.org/l/25Addsubtrpoly3)**
- **Add and Subtract Polynomial 4 (https://openstax.org/l/25Addsubtrpoly4)**

6.1 EXERCISES

Practice Makes Perfect

Identify Polynomials, Monomials, Binomials, and Trinomials

In the following exercises, determine if each of the following polynomials is a monomial, binomial, trinomial, or other polynomial.

1.
ⓐ $81b^5 - 24b^3 + 1$
ⓑ $5c^3 + 11c^2 - c - 8$
ⓒ $\frac{14}{15}y + \frac{1}{7}$
ⓓ 5
ⓔ $4y + 17$

2.
ⓐ $x^2 - y^2$
ⓑ $-13c^4$
ⓒ $x^2 + 5x - 7$
ⓓ $x^2y^2 - 2xy + 8$
ⓔ 19

3.
ⓐ $8 - 3x$
ⓑ $z^2 - 5z - 6$
ⓒ $y^3 - 8y^2 + 2y - 16$
ⓓ $81b^5 - 24b^3 + 1$
ⓔ -18

4.
ⓐ $11y^2$
ⓑ -73
ⓒ $6x^2 - 3xy + 4x - 2y + y^2$
ⓓ $4y + 17$
ⓔ $5c^3 + 11c^2 - c - 8$

Determine the Degree of Polynomials

In the following exercises, determine the degree of each polynomial.

5.
ⓐ $6a^2 + 12a + 14$
ⓑ $18xy^2z$
ⓒ $5x + 2$
ⓓ $y^3 - 8y^2 + 2y - 16$
ⓔ -24

6.
ⓐ $9y^3 - 10y^2 + 2y - 6$
ⓑ $-12p^4$
ⓒ $a^2 + 9a + 18$
ⓓ $20x^2y^2 - 10a^2b^2 + 30$
ⓔ 17

7.
ⓐ $14 - 29x$
ⓑ $z^2 - 5z - 6$
ⓒ $y^3 - 8y^2 + 2y - 16$
ⓓ $23ab^2 - 14$
ⓔ -3

8.
ⓐ $62y^2$
ⓑ 15
ⓒ $6x^2 - 3xy + 4x - 2y + y^2$
ⓓ $10 - 9x$
ⓔ $m^4 + 4m^3 + 6m^2 + 4m + 1$

Add and Subtract Monomials

In the following exercises, add or subtract the monomials.

9. $7x^2 + 5x^2$

10. $4y^3 + 6y^3$

11. $-12w + 18w$

12. $-3m + 9m$

13. $4a - 9a$

14. $-y - 5y$

15. $28x - (-12x)$

16. $13z - (-4z)$

17. $-5b - 17b$

18. $-10x - 35x$

19. $12a + 5b - 22a$

20. $14\text{x} - 3y - 13x$

21. $2a^2 + b^2 - 6a^2$

22. $5u^2 + 4v^2 - 6u^2$

23. $xy^2 - 5x - 5y^2$

24. $pq^2 - 4p - 3q^2$

25. $a^2 b - 4a - 5ab^2$

26. $x^2 y - 3x + 7xy^2$

27. $12\text{a} + 8b$

28. $19\text{y} + 5z$

29. Add: $4a,\ -3b,\ -8a$

30. Add: $4\text{x},\ 3y,\ -3x$

31. Subtract $5x^6$ from $-12x^6$.

32. Subtract $2p^4$ from $-7p^4$.

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

33. $(5y^2 + 12y + 4) + (6y^2 - 8y + 7)$

34. $(4y^2 + 10y + 3) + (8y^2 - 6y + 5)$

35. $(x^2 + 6x + 8) + (-4x^2 + 11x - 9)$

36. $(y^2 + 9y + 4) + (-2y^2 - 5y - 1)$

37. $(8x^2 - 5x + 2) + (3x^2 + 3)$

38. $(7x^2 - 9x + 2) + (6x^2 - 4)$

39. $(5a^2 + 8) + (a^2 - 4a - 9)$

40. $(p^2 - 6p - 18) + (2p^2 + 11)$

41. $(4m^2 - 6m - 3) - (2m^2 + m - 7)$

42. $(3b^2 - 4b + 1) - (5b^2 - b - 2)$

43. $(a^2 + 8a + 5) - (a^2 - 3a + 2)$

44. $(b^2 - 7b + 5) - (b^2 - 2b + 9)$

45. $(12s^2 - 15s) - (s - 9)$

46. $(10r^2 - 20r) - (r - 8)$

47. Subtract $\left(9x^2 + 2\right)$ from $\left(12x^2 - x + 6\right)$.

48. Subtract $\left(5y^2 - y + 12\right)$ from $\left(10y^2 - 8y - 20\right)$.

49. Subtract $\left(7w^2 - 4w + 2\right)$ from $\left(8w^2 - w + 6\right)$.

50. Subtract $\left(5x^2 - x + 12\right)$ from $\left(9x^2 - 6x - 20\right)$.

51. Find the sum of $\left(2p^3 - 8\right)$ and $\left(p^2 + 9p + 18\right)$.

52. Find the sum of $\left(q^2 + 4q + 13\right)$ and $\left(7q^3 - 3\right)$.

53. Find the sum of $\left(8a^3 - 8a\right)$ and $\left(a^2 + 6a + 12\right)$.

54. Find the sum of $\left(b^2 + 5b + 13\right)$ and $\left(4b^3 - 6\right)$.

55. Find the difference of $\left(w^2 + w - 42\right)$ and $\left(w^2 - 10w + 24\right)$.

56. Find the difference of $\left(z^2 - 3z - 18\right)$ and $\left(z^2 + 5z - 20\right)$.

57. Find the difference of $\left(c^2 + 4c - 33\right)$ and $\left(c^2 - 8c + 12\right)$.

58. Find the difference of $\left(t^2 - 5t - 15\right)$ and $\left(t^2 + 4t - 17\right)$.

59. $\left(7x^2 - 2xy + 6y^2\right) + \left(3x^2 - 5xy\right)$

60. $\left(-5x^2 - 4xy - 3y^2\right) + \left(2x^2 - 7xy\right)$

61. $\left(7m^2 + mn - 8n^2\right) + \left(3m^2 + 2mn\right)$

62. $\left(2r^2 - 3rs - 2s^2\right) + \left(5r^2 - 3rs\right)$

63. $\left(a^2 - b^2\right) - \left(a^2 + 3ab - 4b^2\right)$

64. $\left(m^2 + 2n^2\right) - \left(m^2 - 8mn - n^2\right)$

65. $\left(u^2 - v^2\right) - \left(u^2 - 4uv - 3v^2\right)$

66. $\left(j^2 - k^2\right) - \left(j^2 - 8jk - 5k^2\right)$

67. $\left(p^3 - 3p^2q\right) + \left(2pq^2 + 4q^3\right) - \left(3p^2q + pq^2\right)$

68. $\left(a^3 - 2a^2b\right) + \left(ab^2 + b^3\right) - \left(3a^2b + 4ab^2\right)$

69. $\left(x^3 - x^2y\right) - \left(4xy^2 - y^3\right) + \left(3x^2y - xy^2\right)$

70. $\left(x^3 - 2x^2y\right) - \left(xy^2 - 3y^3\right) - \left(x^2y - 4xy^2\right)$

Evaluate a Polynomial for a Given Value

In the following exercises, evaluate each polynomial for the given value.

71. Evaluate $8y^2 - 3y + 2$ when:

ⓐ $y = 5$
ⓑ $y = -2$
ⓒ $y = 0$

72. Evaluate $5y^2 - y - 7$ when:

ⓐ $y = -4$
ⓑ $y = 1$
ⓒ $y = 0$

73. Evaluate $4 - 36x$ when:

ⓐ $x = 3$
ⓑ $x = 0$
ⓒ $x = -1$

74. Evaluate $16 - 36x^2$ when:

ⓐ $x = -1$
ⓑ $x = 0$
ⓒ $x = 2$

75. A painter drops a brush from a platform 75 feet high. The polynomial $-16t^2 + 75$ gives the height of the brush t seconds after it was dropped. Find the height after $t = 2$ seconds.

76. A girl drops a ball off a cliff into the ocean. The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall cliff. Find the height after $t = 2$ seconds.

77. A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of *p* dollars each is given by the polynomial $-4p^2 + 420p$. Find the revenue received when $p = 60$ dollars.

78. A manufacturer of the latest basketball shoes has found that the revenue received from selling the shoes at a cost of *p* dollars each is given by the polynomial $-4p^2 + 420p$. Find the revenue received when $p = 90$ dollars.

Everyday Math

79. Fuel Efficiency The fuel efficiency (in miles per gallon) of a car going at a speed of x miles per hour is given by the polynomial $-\frac{1}{150}x^2 + \frac{1}{3}x$. Find the fuel efficiency when $x = 30 \text{ mph}$.

80. Stopping Distance The number of feet it takes for a car traveling at x miles per hour to stop on dry, level concrete is given by the polynomial $0.06x^2 + 1.1x$. Find the stopping distance when $x = 40 \text{ mph}$.

81. Rental Cost The cost to rent a rug cleaner for d days is given by the polynomial $5.50d + 25$. Find the cost to rent the cleaner for 6 days.

82. Height of Projectile The height (in feet) of an object projected upward is given by the polynomial $-16t^2 + 60t + 90$ where t represents time in seconds. Find the height after $t = 2.5$ seconds.

83. Temperature Conversion The temperature in degrees Fahrenheit is given by the polynomial $\frac{9}{5}c + 32$ where c represents the temperature in degrees Celsius. Find the temperature in degrees Fahrenheit when $c = 65°$.

Writing Exercises

84. Using your own words, explain the difference between a monomial, a binomial, and a trinomial.

85. Using your own words, explain the difference between a polynomial with five terms and a polynomial with a degree of 5.

86. Ariana thinks the sum $6y^2 + 5y^4$ is $11y^6$. What is wrong with her reasoning?

87. Jonathan thinks that $\frac{1}{3}$ and $\frac{1}{x}$ are both monomials. What is wrong with his reasoning?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
identify polynomials, monomials, binomials, and trinomials.			
determine the degree of polynomials.			
add and subtract monomials.			
add and subtract polynomials.			
evaluate a polynomial for a given value.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

...with some help. *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

6.2 Use Multiplication Properties of Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with exponents
- Simplify expressions using the Product Property for Exponents
- Simplify expressions using the Power Property for Exponents
- Simplify expressions using the Product to a Power Property
- Simplify expressions by applying several properties
- Multiply monomials

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{3}{4}\cdot\frac{3}{4}$.
   If you missed this problem, review **Example 1.68**.
2. Simplify: $(-2)(-2)(-2)$.
   If you missed this problem, review **Example 1.50**.

Simplify Expressions with Exponents

Remember that an exponent indicates repeated multiplication of the same quantity. For example, 2^4 means to multiply 2 by itself 4 times, so 2^4 means $2\cdot 2\cdot 2\cdot 2$.

Let's review the vocabulary for expressions with exponents.

Exponential Notation

a^m means multiply m factors of a

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots\ldots \cdot a}_{m\text{ factors}}$$

This is read a to the m^{th} power.

In the expression a^m, the *exponent* m tells us how many times we use the *base* a as a factor.

$$4^3 \qquad\qquad (-9)^5$$

$$\underbrace{4\cdot 4\cdot 4}_{3\text{ factors}} \qquad\qquad \underbrace{(-9)(-9)(-9)(-9)(-9)}_{5\text{ factors}}$$

Before we begin working with variable expressions containing exponents, let's simplify a few expressions involving only numbers.

EXAMPLE 6.16

Simplify: ⓐ 4^3 ⓑ 7^1 ⓒ $\left(\frac{5}{6}\right)^2$ ⓓ $(0.63)^2$.

Solution

ⓐ

	4^3
Multiply three factors of 4.	$4 \cdot 4 \cdot 4$
Simplify.	64

ⓑ

	7^1
Multiply one factor of 7.	7

ⓒ

	$\left(\frac{5}{6}\right)^2$
Multiply two factors.	$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)$
Simplify.	$\frac{25}{36}$

ⓓ

	$(0.63)^2$
Multiply two factors.	$(0.63)(0.63)$
Simplify.	0.3969

TRY IT : : 6.31 Simplify: ⓐ 6^3 ⓑ 15^1 ⓒ $\left(\frac{3}{7}\right)^2$ ⓓ $(0.43)^2$.

TRY IT : : 6.32 Simplify: ⓐ 2^5 ⓑ 21^1 ⓒ $\left(\frac{2}{5}\right)^3$ ⓓ $(0.218)^2$.

EXAMPLE 6.17

Simplify: ⓐ $(-5)^4$ ⓑ -5^4.

Solution

ⓐ

	$(-5)^4$
Multiply four factors of -5.	$(-5)(-5)(-5)(-5)$
Simplify.	625

ⓑ

	-5^4
Multiply four factors of 5.	$-(5 \cdot 5 \cdot 5 \cdot 5)$
Simplify.	-625

TRY IT : : 6.33 Simplify: ⓐ $(-3)^4$ ⓑ -3^4.

TRY IT : : 6.34 Simplify: ⓐ $(-13)^2$ ⓑ -13^2.

Notice the similarities and differences in **Example 6.17**ⓐ and **Example 6.17**ⓑ! Why are the answers different? As we follow the order of operations in part ⓐ the parentheses tell us to raise the (-5) to the 4th power. In part ⓑ we raise just the 5 to the 4th power and then take the opposite.

Simplify Expressions Using the Product Property for Exponents

You have seen that when you combine like terms by adding and subtracting, you need to have the same base with the same exponent. But when you multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

We'll derive the properties of exponents by looking for patterns in several examples.

First, we will look at an example that leads to the Product Property.

	$x^2 \cdot x^3$
What does this mean? How many factors altogether?	$\underbrace{\underbrace{x \cdot x}_{2 \text{ factors}} \cdot \underbrace{x \cdot x \cdot x}_{3 \text{ factors}}}_{5 \text{ factors}}$
So, we have	x^5
Notice that 5 is the sum of the exponents, 2 and 3.	$x^2 \cdot x^3$ is x^{2+3}, or x^5

We write:

$$x^2 \cdot x^3$$
$$x^{2+3}$$
$$x^5$$

The base stayed the same and we added the exponents. This leads to the **Product Property for Exponents**.

Product Property for Exponents

If a is a real number, and m and n are counting numbers, then

$$a^m \cdot a^n = a^{m+n}$$

To multiply with like bases, add the exponents.

An example with numbers helps to verify this property.

$$\begin{aligned} 2^2 \cdot 2^3 &\stackrel{?}{=} 2^{2+3} \\ 4 \cdot 8 &\stackrel{?}{=} 2^5 \\ 32 &= 32 \checkmark \end{aligned}$$

EXAMPLE 6.18

Simplify: $y^5 \cdot y^6$.

Solution

	$y^5 \cdot y^6$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	y^{5+6}
Simplify.	y^{11}

TRY IT : : 6.35 Simplify: $b^9 \cdot b^8$.

TRY IT : : 6.36 Simplify: $x^{12} \cdot x^4$.

EXAMPLE 6.19

Simplify: ⓐ $2^5 \cdot 2^9$ ⓑ $3 \cdot 3^4$.

Solution

ⓐ

	$2^5 \cdot 2^9$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	2^{5+9}
Simplify.	2^{14}

ⓑ

	$3 \cdot 3^4$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	3^{1+4}
Simplify.	3^5

TRY IT : : 6.37 Simplify: ⓐ $5 \cdot 5^5$ ⓑ $4^9 \cdot 4^9$.

TRY IT : : 6.38 Simplify: ⓐ $7^6 \cdot 7^8$ ⓑ $10 \cdot 10^{10}$.

EXAMPLE 6.20

Simplify: ⓐ $a^7 \cdot a$ ⓑ $x^{27} \cdot x^{13}$.

Solution

ⓐ

	$a^7 \cdot a$
Rewrite, $a = a^1$.	$a^7 \cdot a^1$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	a^{7+1}
Simplify.	a^8

ⓑ

	$x^{27} \cdot x^{13}$
Notice, the bases are the same, so add the exponents.	x^{27+13}
Simplify.	x^{40}

TRY IT : : 6.39 Simplify: ⓐ $p^5 \cdot p$ ⓑ $y^{14} \cdot y^{29}$.

TRY IT : : 6.40 Simplify: ⓐ $z \cdot z^7$ ⓑ $b^{15} \cdot b^{34}$.

We can extend the Product Property for Exponents to more than two factors.

EXAMPLE 6.21

Simplify: $d^4 \cdot d^5 \cdot d^2$.

Solution

	$d^4 \cdot d^5 \cdot d^2$
Add the exponents, since bases are the same.	d^{4+5+2}
Simplify.	d^{11}

TRY IT : : 6.41 Simplify: $x^6 \cdot x^4 \cdot x^8$.

TRY IT : : 6.42 Simplify: $b^5 \cdot b^9 \cdot b^5$.

Simplify Expressions Using the Power Property for Exponents

Now let's look at an exponential expression that contains a power raised to a power. See if you can discover a general property.

	$(x^2)^3$
What does this mean? How many factors altogether?	$x^2 \cdot x^2 \cdot x^2$ $\underbrace{\underbrace{x \cdot x}_{2\textit{ factors}} \cdot \underbrace{x \cdot x}_{2\textit{ factors}} \cdot \underbrace{x \cdot x}_{2\textit{ factors}}}_{6\textit{ factors}}$
So we have	x^6
Notice that 6 is the product of the exponents, 2 and 3.	$(x^2)^3$ is $x^{2\cdot 3}$ or x^6

We write:

$$\left(x^2\right)^3$$
$$x^{2\cdot 3}$$
$$x^6$$

We multiplied the exponents. This leads to the **Power Property for Exponents.**

Power Property for Exponents

If a is a real number, and m and n are whole numbers, then

$$(a^m)^n = a^{m\cdot n}$$

To raise a power to a power, multiply the exponents.

An example with numbers helps to verify this property.

$$\begin{aligned} \left(3^2\right)^3 &\stackrel{?}{=} 3^{2\cdot 3} \\ (9)^3 &\stackrel{?}{=} 3^6 \\ 729 &= 729 ✓ \end{aligned}$$

EXAMPLE 6.22

Simplify: ⓐ $(y^5)^9$ ⓑ $(4^4)^7$.

Solution

ⓐ

	$(y^5)^9$
Use the power property, $(a^m)^n = a^{m\cdot n}$.	$y^{5\cdot 9}$
Simplify.	y^{45}

ⓑ

	$(4^4)^7$
Use the power property.	$4^{4\cdot 7}$
Simplify.	4^{28}

> TRY IT : : 6.43 Simplify: ⓐ $(b^7)^5$ ⓑ $(5^4)^3$.

> TRY IT : : 6.44 Simplify: ⓐ $(z^6)^9$ ⓑ $(3^7)^7$.

Simplify Expressions Using the Product to a Power Property

We will now look at an expression containing a product that is raised to a power. Can you find this pattern?

	$(2x)^3$
What does this mean?	$2x\cdot 2x\cdot 2x$
We group the like factors together.	$2\cdot 2\cdot 2\cdot x\cdot x\cdot x$
How many factors of 2 and of x?	$2^3\cdot x^3$

Notice that each factor was raised to the power and $(2x)^3$ is $2^3\cdot x^3$.

We write:	$(2x)^3$
	$2^3\cdot x^3$

The exponent applies to each of the factors! This leads to the **Product to a Power Property for Exponents.**

Product to a Power Property for Exponents

If a and b are real numbers and m is a whole number, then

$$(ab)^m = a^m b^m$$

To raise a product to a power, raise each factor to that power.

An example with numbers helps to verify this property:

$$\begin{aligned}(2\cdot 3)^2 &\stackrel{?}{=} 2^2\cdot 3^2\\ 6^2 &\stackrel{?}{=} 4\cdot 9\\ 36 &= 36\,\checkmark\end{aligned}$$

EXAMPLE 6.23

Simplify: ⓐ $(-9d)^2$ ⓑ $(3mn)^3$.

Solution

ⓐ

	$(-9d)^2$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(-9)^2 d^2$
Simplify.	$81d^2$

ⓑ

	$(3mn)^3$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(3)^3 m^3 n^3$
Simplify.	$27m^3n^3$

TRY IT : : 6.45 Simplify: ⓐ $(-12y)^2$ ⓑ $(2wx)^5$.

TRY IT : : 6.46 Simplify: ⓐ $(5wx)^3$ ⓑ $(-3y)^3$.

Simplify Expressions by Applying Several Properties

We now have three properties for multiplying expressions with exponents. Let's summarize them and then we'll do some examples that use more than one of the properties.

Properties of Exponents

If a and b are real numbers, and m and n are whole numbers, then

$$\begin{array}{lrcl} \textbf{Product Property} & a^m \cdot a^n & = & a^{m+n} \\ \textbf{Power Property} & (a^m)^n & = & a^{m \cdot n} \\ \textbf{Product to a Power} & (ab)^m & = & a^m b^m \end{array}$$

All exponent properties hold true for any real numbers m and n. Right now, we only use whole number exponents.

EXAMPLE 6.24

Simplify: ⓐ $(y^3)^6 (y^5)^4$ ⓑ $\left(-6x^4 y^5\right)^2$.

Solution

ⓐ

	$(y^3)^6 (y^5)^4$
Use the Power Property.	$y^{15} \cdot y^{20}$
Add the exponents.	y^{35}

ⓑ

	$\left(-6x^4 y^5\right)^2$
Use the Product to a Power Property.	$(-6)^2 (x^4)^2 (y^5)^2$
Use the Power Property.	$(-6)^2 (x^8)(y^{10})$
Simplify.	$36x^8 y^{10}$

TRY IT : : 6.47 Simplify: ⓐ $(a^4)^5 (a^7)^4$ ⓑ $\left(-2c^4 d^2\right)^3$.

TRY IT : : 6.48 Simplify: ⓐ $\left(-3x^6 y^7\right)^4$ ⓑ $(q^4)^5 (q^3)^3$.

EXAMPLE 6.25

Simplify: ⓐ $(5m)^2(3m^3)$ ⓑ $(3x^2y)^4(2xy^2)^3$.

Solution

ⓐ

	$(5m)^2(3m^3)$
Raise $5m$ to the second power.	$5^2m^2 \cdot 3m^3$
Simplify.	$25m^2 \cdot 3m^3$
Use the Commutative Property.	$25 \cdot 3 \cdot m^2 \cdot m^3$
Multiply the constants and add the exponents.	$75m^5$

ⓑ

	$(3x^2y)^4(2xy^2)^3$
Use the Product to a Power Property.	$(3^4x^8y^4)(2^3x^3y^6)$
Simplify.	$(81x^8y^4)(8x^3y^6)$
Use the Commutative Property.	$81 \cdot 8 \cdot x^8 \cdot x^3 \cdot y^4 \cdot y^6$
Multiply the constants and add the exponents.	$648x^{11}y^{10}$

TRY IT : : 6.49 Simplify: ⓐ $(5n)^2(3n^{10})$ ⓑ $(c^4d^2)^5(3cd^5)^4$.

TRY IT : : 6.50 Simplify: ⓐ $(a^3b^2)^6(4ab^3)^4$ ⓑ $(2x)^3(5x^7)$.

Multiply Monomials

Since a monomial is an algebraic expression, we can use the properties of exponents to multiply monomials.

EXAMPLE 6.26

Multiply: $(3x^2)(-4x^3)$.

Solution

	$(3x^2)(-4x^3)$
Use the Commutative Property to rearrange the terms.	$3 \cdot (-4) \cdot x^2 \cdot x^3$
Multiply.	$-12x^5$

TRY IT : : 6.51 Multiply: $(5y^7)(-7y^4)$.

TRY IT : : 6.52 Multiply: $(-6b^4)(-9b^5)$.

EXAMPLE 6.27

Multiply: $\left(\frac{5}{6}x^3y\right)(12xy^2)$.

Solution

	$\left(\frac{5}{6}x^3 y\right)\left(12xy^2\right)$
Use the Commutative Property to rearrange the terms.	$\frac{5}{6} \cdot 12 \cdot x^3 \cdot x \cdot y \cdot y^2$
Multiply.	$10x^4 y^3$

TRY IT : : 6.53 Multiply: $\left(\frac{2}{5}a^4 b^3\right)\left(15ab^3\right)$.

TRY IT : : 6.54 Multiply: $\left(\frac{2}{3}r^5 s\right)\left(12r^6 s^7\right)$.

MEDIA : :

Access these online resources for additional instruction and practice with using multiplication properties of exponents:

- **Multiplication Properties of Exponents (https://openstax.org/l/25MultiPropExp)**

6.2 EXERCISES

Practice Makes Perfect

Simplify Expressions with Exponents

In the following exercises, simplify each expression with exponents.

88.
ⓐ 3^5
ⓑ 9^1
ⓒ $\left(\frac{1}{3}\right)^2$
ⓓ $(0.2)^4$

89.
ⓐ 10^4
ⓑ 17^1
ⓒ $\left(\frac{2}{9}\right)^2$
ⓓ $(0.5)^3$

90.
ⓐ 2^6
ⓑ 14^1
ⓒ $\left(\frac{2}{5}\right)^3$
ⓓ $(0.7)^2$

91.
ⓐ 8^3
ⓑ 8^1
ⓒ $\left(\frac{3}{4}\right)^3$
ⓓ $(0.4)^3$

92.
ⓐ $(-6)^4$
ⓑ -6^4

93.
ⓐ $(-2)^6$
ⓑ -2^6

94.
ⓐ $-\left(\frac{1}{4}\right)^4$
ⓑ $\left(-\frac{1}{4}\right)^4$

95.
ⓐ $-\left(\frac{2}{3}\right)^2$
ⓑ $\left(-\frac{2}{3}\right)^2$

96.
ⓐ -0.5^2
ⓑ $(-0.5)^2$

97.
ⓐ -0.1^4
ⓑ $(-0.1)^4$

Simplify Expressions Using the Product Property for Exponents

In the following exercises, simplify each expression using the Product Property for Exponents.

98. $d^3 \cdot d^6$

99. $x^4 \cdot x^2$

100. $n^{19} \cdot n^{12}$

101. $q^{27} \cdot q^{15}$

102. ⓐ $4^5 \cdot 4^9$ ⓑ $8^9 \cdot 8$

103. ⓐ $3^{10} \cdot 3^6$ ⓑ $5 \cdot 5^4$

104. ⓐ $y \cdot y^3$ ⓑ $z^{25} \cdot z^8$

105. ⓐ $w^5 \cdot w$ ⓑ $u^{41} \cdot u^{53}$

106. $w \cdot w^2 \cdot w^3$

107. $y \cdot y^3 \cdot y^5$

108. $a^4 \cdot a^3 \cdot a^9$

109. $c^5 \cdot c^{11} \cdot c^2$

110. $m^x \cdot m^3$

111. $n^y \cdot n^2$

112. $y^a \cdot y^b$

113. $x^p \cdot x^q$

Simplify Expressions Using the Power Property for Exponents

In the following exercises, simplify each expression using the Power Property for Exponents.

114. ⓐ $(m^4)^2$ ⓑ $(10^3)^6$

115. ⓐ $(b^2)^7$ ⓑ $(3^8)^2$

116. ⓐ $\left(y^3\right)^x$ ⓑ $(5^x)^y$

117. ⓐ $\left(x^2\right)^y$ ⓑ $(7^a)^b$

Simplify Expressions Using the Product to a Power Property

In the following exercises, simplify each expression using the Product to a Power Property.

118. ⓐ $(6a)^2$ ⓑ $(3xy)^2$

119. ⓐ $(5x)^2$ ⓑ $(4ab)^2$

120. ⓐ $(-4m)^3$ ⓑ $(5ab)^3$

121. ⓐ $(-7n)^3$ ⓑ $(3xyz)^4$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify each expression.

122.
ⓐ $\left(y^2\right)^4 \cdot \left(y^3\right)^2$
ⓑ $(10a^2b)^3$

123.
ⓐ $\left(w^4\right)^3 \cdot \left(w^5\right)^2$
ⓑ $(2xy^4)^5$

124.
ⓐ $(-2r^3s^2)^4$
ⓑ $\left(m^5\right)^3 \cdot \left(m^9\right)^4$

125.
ⓐ $(-10q^2p^4)^3$
ⓑ $\left(n^3\right)^{10} \cdot \left(n^5\right)^2$

126.
ⓐ $(3x)^2(5x)$
ⓑ $(5t^2)^3(3t)^2$

127.
ⓐ $(2y)^3(6y)$
ⓑ $(10k^4)^3(5k^6)^2$

128.
ⓐ $(5a)^2(2a)^3$
ⓑ $(\frac{1}{2}y^2)^3(\frac{2}{3}y)^2$

129.
ⓐ $(4b)^2(3b)^3$
ⓑ $(\frac{1}{2}j^2)^5(\frac{2}{5}j^3)^2$

130.
ⓐ $\left(\frac{2}{5}x^2y\right)^3$
ⓑ $\left(\frac{8}{9}xy^4\right)^2$

131.
ⓐ $(2r^2)^3(4r)^2$
ⓑ $(3x^3)^3(x^5)^4$

132.
ⓐ $(m^2n)^2(2mn^5)^4$
ⓑ $(3pq^4)^2(6p^6q)^2$

Multiply Monomials

In the following exercises, multiply the monomials.

133. $\left(6y^7\right)\left(-3y^4\right)$

134. $\left(-10x^5\right)\left(-3x^3\right)$

135. $(-8u^6)(-9u)$

136. $(-6c^4)(-12c)$

137. $(\frac{1}{5}f^8)(20f^3)$

138. $(\frac{1}{4}d^5)(36d^2)$

139. $(4a^3b)(9a^2b^6)$

140. $(6m^4n^3)(7mn^5)$

141. $\left(\frac{4}{7}rs^2\right)\left(14rs^3\right)$

142. $\left(\frac{5}{8}x^3y\right)\left(24x^5y\right)$

143. $\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)$

144. $\left(\frac{3}{5}m^3n^2\right)\left(\frac{5}{9}m^2n^3\right)$

Mixed Practice

In the following exercises, simplify each expression.

145. $\left(x^2\right)^4 \cdot \left(x^3\right)^2$

146. $\left(y^4\right)^3 \cdot \left(y^5\right)^2$

147. $\left(a^2\right)^6 \cdot \left(a^3\right)^8$

148. $\left(b^7\right)^5 \cdot \left(b^2\right)^6$

149. $(2m^6)^3$

150. $(3y^2)^4$

151. $(10x^2y)^3$

152. $(2mn^4)^5$

153. $(-2a^3b^2)^4$

154. $(-10u^2v^4)^3$

155. $\left(\frac{2}{3}x^2y\right)^3$

156. $\left(\frac{7}{9}pq^4\right)^2$

157. $(8a^3)^2(2a)^4$

158. $(5r^2)^3(3r)^2$

159. $(10p^4)^3(5p^6)^2$

160. $(4x^3)^3(2x^5)^4$

161. $\left(\frac{1}{2}x^2y^3\right)^4\left(4x^5y^3\right)^2$

162. $\left(\frac{1}{3}m^3n^2\right)^4\left(9m^8n^3\right)^2$

163. $(3m^2n)^2(2mn^5)^4$

164. $(2pq^4)^3(5p^6q)^2$

Everyday Math

165. Email Kate emails a flyer to ten of her friends and tells them to forward it to ten of their friends, who forward it to ten of their friends, and so on. The number of people who receive the email on the second round is 10^2, on the third round is 10^3, as shown in the table below. How many people will receive the email on the sixth round? Simplify the expression to show the number of people who receive the email.

Round	Number of people
1	10
2	10^2
3	10^3
…	…
6	?

166. Salary Jamal's boss gives him a 3% raise every year on his birthday. This means that each year, Jamal's salary is 1.03 times his last year's salary. If his original salary was \$35,000, his salary after 1 year was $\$35{,}000(1.03)$, after 2 years was $\$35{,}000(1.03)^2$, after 3 years was $\$35{,}000(1.03)^3$, as shown in the table below. What will Jamal's salary be after 10 years? Simplify the expression, to show Jamal's salary in dollars.

Year	Salary
1	$\$35{,}000(1.03)$
2	$\$35{,}000(1.03)^2$
3	$\$35{,}000(1.03)^3$
…	…
10	?

167. Clearance A department store is clearing out merchandise in order to make room for new inventory. The plan is to mark down items by 30% each week. This means that each week the cost of an item is 70% of the previous week's cost. If the original cost of a sofa was $1,000, the cost for the first week would be $\$1{,}000(0.70)$ and the cost of the item during the second week would be $\$1{,}000(0.70)^2$. Complete the table shown below. What will be the cost of the sofa during the fifth week? Simplify the expression, to show the cost in dollars.

Week	Cost
1	$\$1{,}000(0.70)$
2	$\$1{,}000(0.70)^2$
3	
...	...
8	?

168. Depreciation Once a new car is driven away from the dealer, it begins to lose value. Each year, a car loses 10% of its value. This means that each year the value of a car is 90% of the previous year's value. If a new car was purchased for $20,000, the value at the end of the first year would be $\$20{,}000(0.90)$ and the value of the car after the end of the second year would be $\$20{,}000(0.90)^2$. Complete the table shown below. What will be the value of the car at the end of the eighth year? Simplify the expression, to show the value in dollars.

Week	Cost
1	$\$20{,}000(0.90)$
2	$\$20{,}000(0.90)^2$
3	
4	...
5	?

Writing Exercises

169. Use the Product Property for Exponents to explain why $x \cdot x = x^2$.

170. Explain why $-5^3 = (-5)^3$ but $-5^4 \neq (-5)^4$.

171. Jorge thinks $\left(\frac{1}{2}\right)^2$ is 1. What is wrong with his reasoning?

172. Explain why $x^3 \cdot x^5$ is x^8, and not x^{15}.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with exponents.			
simplify expressions using the Product Property for Exponents.			
simplify expressions using the Power Property for Exponents.			
simplify expressions using the Product to a Power Property.			
simplify expressions by applying several properties.			
multiply monomials.			

ⓑ *After reviewing this checklist, what will you do to become confident for all goals?*

6.3 Multiply Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a trinomial by a binomial

Be Prepared!

Before you get started, take this readiness quiz.

1. Distribute: $2(x+3)$.
   If you missed this problem, review Example 1.132.
2. Combine like terms: $x^2+9x+7x+63$.
   If you missed this problem, review Example 1.24.

Multiply a Polynomial by a Monomial

We have used the Distributive Property to simplify expressions like $2(x-3)$. You multiplied both terms in the parentheses, $x \text{ and } 3$, by 2, to get $2x-6$. With this chapter's new vocabulary, you can say you were multiplying a binomial, $x-3$, by a monomial, 2.

Multiplying a binomial by a monomial is nothing new for you! Here's an example:

EXAMPLE 6.28

Multiply: $4(x+3)$.

Solution

	$4(x+3)$
Distribute.	$4 \cdot x + 4 \cdot 3$
Simplify.	$4x+12$

TRY IT : : 6.55 Multiply: $5(x+7)$.

TRY IT : : 6.56 Multiply: $3(y+13)$.

EXAMPLE 6.29

Multiply: $y(y-2)$.

Solution

	$y(y-2)$
Distribute.	$y \cdot y - y \cdot 2$
Simplify.	y^2-2y

TRY IT : : 6.57 Multiply: $x(x-7)$.

TRY IT : : 6.58 Multiply: $d(d-11)$.

EXAMPLE 6.30

Multiply: $7x(2x+y)$.

Solution

	$7x(2x+y)$
Distribute.	$7x \cdot 2x + 7x \cdot y$
Simplify.	$14x^2 + 7xy$

TRY IT : : 6.59 Multiply: $5x(x+4y)$.

TRY IT : : 6.60 Multiply: $2p(6p+r)$.

EXAMPLE 6.31

Multiply: $-2y\left(4y^2+3y-5\right)$.

Solution

	$-2y(4y^2+3y-5)$
Distribute.	$-2y \cdot 4y^2 + (-2y) \cdot 3y - (-2y) \cdot 5$
Simplify.	$-8y^3 - 6y^2 + 10y$

TRY IT : : 6.61 Multiply: $-3y\left(5y^2+8y-7\right)$.

TRY IT : : 6.62 Multiply: $4x^2(2x^2-3x+5)$.

EXAMPLE 6.32

Multiply: $2x^3(x^2-8x+1)$.

Solution

	$2x^3(x^2 - 8x + 1)$
Distribute.	$2x^3 \cdot x^2 + (2x^3) \cdot (-8x) + (2x^3) \cdot 1$
Simplify.	$2x^5 - 16x^4 + 2x^3$

TRY IT : : 6.63 Multiply: $4x(3x^2 - 5x + 3)$.

TRY IT : : 6.64 Multiply: $-6a^3(3a^2 - 2a + 6)$.

EXAMPLE 6.33

Multiply: $(x + 3)p$.

Solution

The monomial is the second factor.	$(x + 3)p$
Distribute.	$x \cdot p + 3 \cdot p$
Simplify.	$xp + 3p$

TRY IT : : 6.65 Multiply: $(x + 8)p$.

TRY IT : : 6.66 Multiply: $(a + 4)p$.

Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial. We will start by using the Distributive Property.

Multiply a Binomial by a Binomial Using the Distributive Property

Look at Example 6.33, where we multiplied a binomial by a monomial.

	$(x+3)p$
We distributed the p to get:	$xp + 3p$
What if we have $(x+7)$ instead of p?	$(x+3)(x+7)$
Distribute $(x+7)$.	$x(x+7) + 3(x+7)$
Distribute again.	$x^2 + 7x + 3x + 21$
Combine like terms.	$x^2 + 10x + 21$

Notice that before combining like terms, you had four terms. You multiplied the two terms of the first binomial by the two terms of the second binomial—four multiplications.

EXAMPLE 6.34

Multiply: $(y+5)(y+8)$.

Solution

	$(y+5)(y+8)$
Distribute $(y+8)$.	$y(y+8) + 5(y+8)$
Distribute again	$y^2 + 8y + 5y + 40$
Combine like terms.	$y^2 + 13y + 40$

TRY IT : : 6.67 Multiply: $(x+8)(x+9)$.

TRY IT : : 6.68 Multiply: $(5x+9)(4x+3)$.

EXAMPLE 6.35

Multiply: $(2y+5)(3y+4)$.

Solution

	$(2y+5)(3y+4)$
Distribute $(3y+4)$.	$2y(3y+4)+5(3y+4)$
Distribute again	$6y^2+8y+15y+20$
Combine like terms.	$6y^2+23y+20$

TRY IT : : 6.69 Multiply: $(3b+5)(4b+6)$.

TRY IT : : 6.70 Multiply: $(a+10)(a+7)$.

EXAMPLE 6.36

Multiply: $(4y+3)(2y-5)$.

Solution

	$(4y+3)(2y-5)$
Distribute.	$4y(2y-5)+3(2y-5)$
Distribute again.	$8y^2-20y+6y-15$
Combine like terms.	$8y^2-14y-15$

TRY IT : : 6.71 Multiply: $(5y+2)(6y-3)$.

TRY IT : : 6.72 Multiply: $(3c+4)(5c-2)$.

EXAMPLE 6.37

Multiply: $(x+2)(x-y)$.

Solution

	$(x-2)(x-y)$
Distribute.	$x(x-y)-2(x-y)$
Distribute again.	$x^2-xy-2x+2y$
There are no like terms to combine.	

TRY IT : : 6.73 Multiply: $(a+7)(a-b)$.

TRY IT : : 6.74 Multiply: $(x+5)(x-y)$.

Multiply a Binomial by a Binomial Using the FOIL Method

Remember that when you multiply a binomial by a binomial you get four terms. Sometimes you can combine like terms to get a trinomial, but sometimes, like in **Example 6.37**, there are no like terms to combine.

Let's look at the last example again and pay particular attention to how we got the four terms.

$$(x-2)(x-y)$$
$$x^2 - xy - 2x + 2y$$

Where did the first term, x^2, come from?

It is the product of x and x, the ***first*** terms in $(x-2)$ and $(x-y)$.	$(x-2)(x-y)$ First
The next term, $-xy$, is the product of x and $-y$, the two ***outer*** terms.	$(x-2)(x-y)$ Outer
The third term, $-2x$, is the product of -2 and x, the two ***inner*** terms.	$(x-2)(x-y)$ Inner
And the last term, $+2y$, came from multiplying the two ***last*** terms, -2 and $-y$.	$(x-2)(x-y)$ Last

We abbreviate "First, Outer, Inner, Last" as FOIL. The letters stand for '**F**irst, **O**uter, **I**nner, **L**ast'. The word FOIL is easy to remember and ensures we find all four products.

$$(x-2)(x-y)$$
$$\underset{F}{x^2} - \underset{O}{xy} - \underset{I}{2x} + \underset{L}{2y}$$

Let's look at $(x+3)(x+7)$.

Distibutive Property	FOIL
$(x+3)(x+7)$	$(x+3)(x+7)$
$x(x+7)+3(x+7)$	
$\underset{F}{x^2} + \underset{O}{7x} + \underset{I}{3x} + \underset{L}{21}$	$\underset{F}{x^2} + \underset{O}{7x} + \underset{I}{3x} + \underset{L}{21}$
$x^2 + 10x + 21$	$x^2 + 10x + 21$

Notice how the terms in third line fit the FOIL pattern.

Now we will do an example where we use the FOIL pattern to multiply two binomials.

EXAMPLE 6.38 HOW TO MULTIPLY A BINOMIAL BY A BINOMIAL USING THE FOIL METHOD

Multiply using the FOIL method: $(x+5)(x+9)$.

Solution

Step 1. Multiply the *First* terms.	$(x+5)(x+9)$ $(x+5)(x+9)$	$x^2 + \_ + \_ + \_$ $F \quad O \quad I \quad L$
Step 2. Multiply the *Outer* terms.	$(x+5)(x+9)$	$x^2 + 9x + \_ + \_$ $F \quad O \quad I \quad L$
Step 3. Multiply the *Inner* terms.	$(x+5)(x+9)$	$x^2 + 9x + 5x + \_$ $F \quad O \quad I \quad L$
Step 4. Multiply the *Last* terms.	$(x+5)(x+9)$	$x^2 + 9x + 5x + 45$ $F \quad O \quad I \quad L$
Step 5. Combine like terms, when possible.		$x^2 + 14x + 45$

TRY IT : : 6.75 Multiply using the FOIL method: $(x+6)(x+8)$.

TRY IT : : 6.76 Multiply using the FOIL method: $(y+17)(y+3)$.

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

HOW TO : : MULTIPLY TWO BINOMIALS USING THE FOIL METHOD

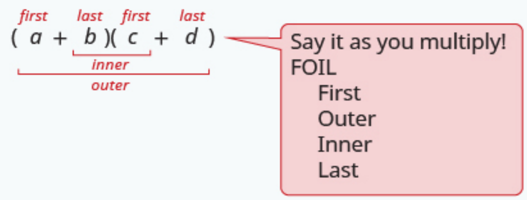
Step 1. Multiply the *First* terms.

Step 2. Multiply the *Outer* terms.

Step 3. Multiply the *Inner* terms.

Step 4. Multiply the *Last* terms.

Step 5. Combine like terms, when possible.



When you multiply by the FOIL method, drawing the lines will help your brain focus on the pattern and make it easier to apply.

EXAMPLE 6.39

Multiply: $(y-7)(y+4)$.

Solution

		$(y-7)(y+4)$
Multiply the *First* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{\_} + \underset{I}{\_} + \underset{L}{\_}$
Multiply the *Outer* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{4y} + \underset{I}{\_} + \underset{L}{\_}$
Multiply the *Inner* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{4y} - \underset{I}{7y} + \underset{L}{\_}$
Multiply the *Last* terms.	$(y-7)(y+4)$	$\underset{F}{y^2} + \underset{O}{4y} - \underset{I}{7y} \underset{L}{-28}$
Combine like terms.		$y^2 - 3y - 28$

TRY IT : : 6.77 Multiply: $(x-7)(x+5)$.

TRY IT : : 6.78 Multiply: $(b-3)(b+6)$.

EXAMPLE 6.40

Multiply: $(4x+3)(2x-5)$.

Solution

	$(4x-3)(2x-5)$
	$(4x+3)(2x-5)$
Multiply the *First* terms, $4x \cdot 2x$.	$\underset{F}{8x^2} + \underset{O}{\_} + \underset{I}{\_} + \underset{L}{\_}$
Multiply the *Outer* terms, $4x \cdot (-5)$.	$\underset{F}{8x^2} \underset{O}{-20x} + \underset{I}{\_} + \underset{L}{\_}$
Multiply the *Inner* terms, $3 \cdot 2x$.	$\underset{F}{8x^2} \underset{O}{-2x} + \underset{I}{6x} + \underset{L}{\_}$
Multiply the *Last* terms, $3 \cdot (-5)$.	$\underset{F}{8x^2} \underset{O}{-20x} + \underset{I}{6x} \underset{L}{-15}$
Combine like terms.	$8x^2 - 14x - 15$

TRY IT : : 6.79 Multiply: $(3x+7)(5x-2)$.

TRY IT : : 6.80 Multiply: $(4y+5)(4y-10)$.

The final products in the last four examples were trinomials because we could combine the two middle terms. This is not always the case.

EXAMPLE 6.41

Multiply: $(3x-y)(2x-5)$.

Solution

	$(3x-y)(2x-5)$
	$(3x-y)(2x-5)$
Multiply the *First*.	$\underset{F}{6x^2}+\underset{O}{\_\_}+\underset{I}{\_\_}+\underset{L}{\_\_}$
Multiply the *Outer*.	$\underset{F}{6x^2}\underset{O}{-15x}+\underset{I}{\_\_}+\underset{L}{\_\_}$
Multiply the *Inner*.	$\underset{F}{6x^2}\underset{O}{-15x}\underset{I}{-2xy}+\underset{L}{\_\_}$
Multiply the *Last*.	$\underset{F}{6x^2}\underset{O}{-15x}\underset{I}{-2xy}\underset{L}{+5y}$
Combine like terms—there are none.	$6x^2-15x-2xy+5y$

TRY IT : : 6.81 Multiply: $(10c-d)(c-6)$.

TRY IT : : 6.82 Multiply: $(7x-y)(2x-5)$.

Be careful of the exponents in the next example.

EXAMPLE 6.42

Multiply: $\left(n^2+4\right)(n-1)$.

Solution

	$(n^2+4)(n-1)$
	$(n^2+4)(n-1)$
Multiply the *First*.	$\underset{F}{n^3}+\underset{O}{\_\_}+\underset{I}{\_\_}+\underset{L}{\_\_}$
Multiply the *Outer*.	$\underset{F}{n^3}\underset{O}{-n^2}+\underset{I}{\_\_}+\underset{L}{\_\_}$
Multiply the *Inner*.	$\underset{F}{n^3}\underset{O}{-n^2}\underset{I}{+4n}+\underset{L}{\_\_}$
Multiply the *Last*.	$\underset{F}{n^3}\underset{O}{-n^2}\underset{I}{+4n}\underset{L}{-4}$
Combine like terms—there are none.	n^3-n^2+4n-4

TRY IT : : 6.83 Multiply: $\left(x^2+6\right)(x-8)$.

TRY IT : : 6.84 Multiply: $\left(y^2+7\right)(y-9)$.

EXAMPLE 6.43

Multiply: $(3pq+5)(6pq-11)$.

Solution

	$(3pq+5)(6pq-11)$	
Multiply the *First*.	$\underset{F}{18p^2q^2} + \underset{O}{\underline{\quad}} + \underset{I}{\underline{\quad}} + \underset{L}{\underline{\quad}}$	$(3pq+5)(6pq-11)$
Multiply the *Outer*.	$\underset{F}{18p^2q^2} \underset{O}{- 33pq} + \underset{I}{\underline{\quad}} + \underset{L}{\underline{\quad}}$	
Multiply the *Inner*.	$\underset{F}{18p^2q^2} \underset{O}{- 33pq} \underset{I}{+ 30pq} + \underset{L}{\underline{\quad}}$	
Multiply the *Last*.	$\underset{F}{18p^2q^2} \underset{O}{- 33pq} \underset{I}{+ 30pq} \underset{L}{- 55}$	
Combine like terms—there are none.	$18p^2q^2 - 3pq - 55$	

TRY IT : : 6.85 Multiply: $(2ab+5)(4ab-4)$.

TRY IT : : 6.86 Multiply: $(2xy+3)(4xy-5)$.

Multiply a Binomial by a Binomial Using the Vertical Method

The FOIL method is usually the quickest method for multiplying two binomials, but it *only* works for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

$$\begin{array}{rll}
23 & & \\
\times 46 & & \\
\hline
138 & \text{partial product} & \text{Start by multiplying 23 by 6 to get 138.} \\
92 & \text{partial product} & \text{Next, multiply 23 by 4, lining up the partial product in the correct columns.} \\
\hline
1058 & \text{product} & \text{Last you add the partial products.}
\end{array}$$

Now we'll apply this same method to multiply two binomials.

EXAMPLE 6.44

Multiply using the Vertical Method: $(3y-1)(2y-6)$.

Solution

It does not matter which binomial goes on the top.

$$\begin{array}{lrl}
\text{Multiply } 3y-1 \text{ by } -6. & 3y-1 & \\
\text{Multiply } 3y-1 \text{ by } 2y. & \times\ 2y-6 & \\
\hline
 & -18y+6 & \text{partial product} \\
 & 6y^2-2y & \text{partial product} \\
\hline
\text{Add like terms.} & 6y^2-20y+6 & \text{product}
\end{array}$$

Notice the partial products are the same as the terms in the FOIL method.

$$\begin{array}{c}(3y-1)(2y-6)\\ 6y^2-2y-18y+6\\ \hline 6y^2-20y+6\end{array} \qquad \begin{array}{r}3y-1\\ \times\ 2y-6\\ \hline -18y+6\\ 6y^2-2y\\ \hline 6y^2-20x+6\end{array}$$

> **TRY IT : :** 6.87 Multiply using the Vertical Method: $(5m-7)(3m-6)$.

> **TRY IT : :** 6.88 Multiply using the Vertical Method: $(6b-5)(7b-3)$.

We have now used three methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. The methods are listed here all together, to help you remember them.

Multiplying Two Binomials

To multiply binomials, use the:

- Distributive Property
- FOIL Method
- Vertical Method

Remember, FOIL only works when multiplying two binomials.

Multiply a Trinomial by a Binomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a trinomial by a binomial. Remember, FOIL will not work in this case, but we can use either the Distributive Property or the Vertical Method. We first look at an example using the Distributive Property.

EXAMPLE 6.45

Multiply using the Distributive Property: $(b+3)(2b^2-5b+8)$.

Solution

	$(b+3)(2b^2-5b+8)$
Distribute.	$b(2b^2-5b+8)+3(2b^2-5b+8)$
Multiply.	$2b^3-5b^2+8b+6b^2-15b+24$
Combine like terms.	$2b^3+b^2-7b+24$

> **TRY IT : :** 6.89 Multiply using the Distributive Property: $(y-3)(y^2-5y+2)$.

> **TRY IT : :** 6.90 Multiply using the Distributive Property: $(x+4)(2x^2-3x+5)$.

Now let's do this same multiplication using the Vertical Method.

EXAMPLE 6.46

Multiply using the Vertical Method: $(b+3)(2b^2-5b+8)$.

Solution

It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Multiply $(2b^2 - 5b + 8)$ by 3.	$2b^2 - 5b + 8$ $\times \quad b + 3$ $6b^2 - 15b + 24$
	$2b^3 - 5b^2 + 8b$
Multiply $(2b^2 - 5b + 8)$ by b.	$2b^3 + b^2 - 7b + 24$
Add like terms.	

TRY IT : : 6.91 Multiply using the Vertical Method: $(y - 3)(y^2 - 5y + 2)$.

TRY IT : : 6.92 Multiply using the Vertical Method: $(x + 4)(2x^2 - 3x + 5)$.

We have now seen two methods you can use to multiply a trinomial by a binomial. After you practice each method, you'll probably find you prefer one way over the other. We list both methods are listed here, for easy reference.

Multiplying a Trinomial by a Binomial

To multiply a trinomial by a binomial, use the:

- Distributive Property
- Vertical Method

MEDIA : :

Access these online resources for additional instruction and practice with multiplying polynomials:

- Multiplying Exponents 1 (https://openstax.org/l/25MultiplyExp1)
- Multiplying Exponents 2 (https://openstax.org/l/25MultiplyExp2)
- Multiplying Exponents 3 (https://openstax.org/l/25MultiplyExp3)

6.3 EXERCISES

Practice Makes Perfect

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

173. $4(w + 10)$

174. $6(b + 8)$

175. $-3(a + 7)$

176. $-5(p + 9)$

177. $2(x - 7)$

178. $7(y - 4)$

179. $-3(k - 4)$

180. $-8(j - 5)$

181. $q(q + 5)$

182. $k(k + 7)$

183. $-b(b + 9)$

184. $-y(y + 3)$

185. $-x(x - 10)$

186. $-p(p - 15)$

187. $6r(4r + s)$

188. $5c(9c + d)$

189. $12x(x - 10)$

190. $9m(m - 11)$

191. $-9a(3a + 5)$

192. $-4p(2p + 7)$

193. $3(p^2 + 10p + 25)$

194. $6(y^2 + 8y + 16)$

195. $-8x(x^2 + 2x - 15)$

196. $-5t(t^2 + 3t - 18)$

197. $5q^3\left(q^3 - 2q + 6\right)$

198. $4x^3\left(x^4 - 3x + 7\right)$

199. $-8y(y^2 + 2y - 15)$

200. $-5m(m^2 + 3m - 18)$

201. $5q^3(q^2 - 2q + 6)$

202. $9r^3(r^2 - 3r + 5)$

203. $-4z^2(3z^2 + 12z - 1)$

204. $-3x^2(7x^2 + 10x - 1)$

205. $(2m - 9)m$

206. $(8j - 1)j$

207. $(w - 6) \cdot 8$

208. $(k - 4) \cdot 5$

209. $4(x + 10)$

210. $6(y + 8)$

211. $15(r - 24)$

212. $12(v - 30)$

213. $-3(m + 11)$

214. $-4(p + 15)$

215. $-8(z - 5)$

216. $-3(x - 9)$

217. $u(u + 5)$

218. $q(q + 7)$

219. $n(n^2 - 3n)$

220. $s(s^2 - 6s)$

221. $6x(4x + y)$

222. $5a(9a + b)$

223. $5p(11p - 5q)$

224. $12u(3u - 4v)$

225. $3(v^2 + 10v + 25)$

226. $6(x^2 + 8x + 16)$

227. $2n(4n^2 - 4n + 1)$

228. $3r(2r^2 - 6r + 2)$

229. $-8y(y^2 + 2y - 15)$

230. $-5m(m^2 + 3m - 18)$

231. $5q^3(q^2 - 2q + 6)$

232. $9r^3(r^2 - 3r + 5)$

233. $-4z^2(3z^2 + 12z - 1)$

234. $-3x^2(7x^2 + 10x - 1)$

235. $(2y - 9)y$

236. $(8b - 1)b$

Multiply a Binomial by a Binomial

In the following exercises, multiply the following binomials using: ⓐ the Distributive Property ⓑ the FOIL method ⓒ the Vertical Method.

237. $(w + 5)(w + 7)$

238. $(y + 9)(y + 3)$

239. $(p + 11)(p - 4)$

240. $(q + 4)(q - 8)$

In the following exercises, multiply the binomials. Use any method.

241. $(x + 8)(x + 3)$

242. $(y + 7)(y + 4)$

243. $(y - 6)(y - 2)$

244. $(x - 7)(x - 2)$

245. $(w - 4)(w + 7)$

246. $(q - 5)(q + 8)$

247. $(p + 12)(p - 5)$

248. $(m + 11)(m - 4)$

249. $(6p + 5)(p + 1)$

250. $(7m + 1)(m + 3)$

251. $(2t - 9)(10t + 1)$

252. $(3r - 8)(11r + 1)$

253. $(5x - y)(3x - 6)$

254. $(10a - b)(3a - 4)$

255. $(a + b)(2a + 3b)$

256. $(r + s)(3r + 2s)$

257. $(4z - y)(z - 6)$

258. $(5x - y)(x - 4)$

259. $(x^2 + 3)(x + 2)$

260. $(y^2 - 4)(y + 3)$

261. $(x^2 + 8)(x^2 - 5)$

262. $(y^2 - 7)(y^2 - 4)$

263. $(5ab - 1)(2ab + 3)$

264. $(2xy + 3)(3xy + 2)$

265. $(6pq - 3)(4pq - 5)$

266. $(3rs - 7)(3rs - 4)$

Multiply a Trinomial by a Binomial

In the following exercises, multiply using ⓐ the Distributive Property ⓑ the Vertical Method.

267. $(x + 5)(x^2 + 4x + 3)$

268. $(u + 4)(u^2 + 3u + 2)$

269. $(y + 8)(4y^2 + y - 7)$

270. $(a + 10)(3a^2 + a - 5)$

In the following exercises, multiply. Use either method.

271. $(w - 7)(w^2 - 9w + 10)$

272. $(p - 4)(p^2 - 6p + 9)$

273. $(3q + 1)(q^2 - 4q - 5)$

274. $(6r + 1)(r^2 - 7r - 9)$

Mixed Practice

275. $(10y - 6) + (4y - 7)$

276. $(15p - 4) + (3p - 5)$

277. $\left(x^2 - 4x - 34\right) - \left(x^2 + 7x - 6\right)$

278. $\left(j^2 - 8j - 27\right) - \left(j^2 + 2j - 12\right)$

279. $5q(3q^2 - 6q + 11)$

280. $8t(2t^2 - 5t + 6)$

281. $(s - 7)(s + 9)$

282. $(x - 5)(x + 13)$

283. $\left(y^2 - 2y\right)(y + 1)$

284. $\left(a^2 - 3a\right)(4a + 5)$

285. $(3n - 4)\left(n^2 + n - 7\right)$

286. $(6k - 1)\left(k^2 + 2k - 4\right)$

287. $(7p + 10)(7p - 10)$

288. $(3y + 8)(3y - 8)$

289. $(4m^2 - 3m - 7)m^2$

290. $(15c^2 - 4c + 5)c^4$

291. $(5a + 7b)(5a + 7b)$

292. $(3x - 11y)(3x - 11y)$

293. $(4y + 12z)(4y - 12z)$

Everyday Math

294. Mental math You can use binomial multiplication to multiply numbers without a calculator. Say you need to multiply 13 times 15. Think of 13 as $10 + 3$ and 15 as $10 + 5$.

ⓐ Multiply $(10 + 3)(10 + 5)$ by the FOIL method.

ⓑ Multiply $13 \cdot 15$ without using a calculator.

ⓒ Which way is easier for you? Why?

295. Mental math You can use binomial multiplication to multiply numbers without a calculator. Say you need to multiply 18 times 17. Think of 18 as $20 - 2$ and 17 as $20 - 3$.

ⓐ Multiply $(20 - 2)(20 - 3)$ by the FOIL method.

ⓑ Multiply $18 \cdot 17$ without using a calculator.

ⓒ Which way is easier for you? Why?

Writing Exercises

296. Which method do you prefer to use when multiplying two binomials: the Distributive Property, the FOIL method, or the Vertical Method? Why?

297. Which method do you prefer to use when multiplying a trinomial by a binomial: the Distributive Property or the Vertical Method? Why?

298. Multiply the following:

$(x + 2)(x - 2)$
$(y + 7)(y - 7)$
$(w + 5)(w - 5)$

Explain the pattern that you see in your answers.

299. Multiply the following:

$(m - 3)(m + 3)$
$(n - 10)(n + 10)$
$(p - 8)(p + 8)$

Explain the pattern that you see in your answers.

300. Multiply the following:

$(p + 3)(p + 3)$
$(q + 6)(q + 6)$
$(r + 1)(r + 1)$

Explain the pattern that you see in your answers.

301. Multiply the following:

$(x - 4)(x - 4)$
$(y - 1)(y - 1)$
$(z - 7)(z - 7)$

Explain the pattern that you see in your answers.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
multiply a polynomial by a monomial.			
multiply a binomial by a binomial.			
multiply a trinomial by a binomial.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

6.4 Special Products

Learning Objectives

By the end of this section, you will be able to:

- Square a binomial using the Binomial Squares Pattern
- Multiply conjugates using the Product of Conjugates Pattern
- Recognize and use the appropriate special product pattern

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: ⓐ 9^2 ⓑ $(-9)^2$ ⓒ -9^2.
   If you missed this problem, review **Example 1.50**.

Square a Binomial Using the Binomial Squares Pattern

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and using the methods of the last section, there is less work to do if you learn to use a pattern.

Let's start by looking at $(x+9)^2$.	
What does this mean?	$(x+9)^2$
It means to multiply $(x+9)$ by itself.	$(x+9)(x+9)$
Then, using FOIL, we get:	$x^2+9x+9x+81$
Combining like terms gives:	$x^2+18x+81$
Here's another one:	$(y-7)^2$
Multiply $(y-7)$ by itself.	$(y-7)(y-7)$
Using FOIL, we get:	$y^2-7y-7y+49$
And combining like terms:	$y^2-14y+49$
And one more:	$(2x+3)^2$
Multiply.	$(2x+3)(2x+3)$
Use FOIL:	$4x^2+6x+6x+9$
Combine like terms.	$4x^2+12x+9$

Look at these results. Do you see any patterns?

What about the number of terms? In each example we squared a binomial and the result was a trinomial.

$$(a+b)^2 = \_\_\_\_ + \_\_\_\_ + \_\_\_\_$$

Now look at the ***first term*** in each result. Where did it come from?

$(x+9)^2$	$(y-7)^2$	$(2x+3)^2$
$(x+9)(x+9)$	$(y-7)(y-7)$	$(2x+3)(2x+3)$
$x^2+9x+9x+81$	$y^2-7y-7y+49$	$4x^2+6x+6x+9$
$x^2+18x+81$	$y^2-14y+49$	$4x^2+12x+9$

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!

$$(a+b)^2 = a^2 + \_\_\_\_ + \_\_\_\_$$

To get the ***first term*** of the product, ***square the first term***.

Where did the ***last term*** come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.

$$(a+b)^2 = \_\_\_\_ + \_\_\_\_ + b^2$$

*To get the **last term** of the product, **square the last term**.*

Finally, look at the ***middle term***. Notice it came from adding the "outer" and the "inner" terms—which are both the same! So the middle term is double the product of the two terms of the binomial.

$$(a+b)^2 = \_\_\_\_ + 2ab + \_\_\_\_$$
$$(a-b)^2 = \_\_\_\_ - 2ab + \_\_\_\_$$

*To get the **middle term** of the product, **multiply the terms and double their product**.*

Putting it all together:

Binomial Squares Pattern

If a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

$$\underbrace{(a+b)^2}_{\text{(binomial)}^2} = \underbrace{a^2}_{\text{(first term)}^2} + \underbrace{2ab}_{\text{2(product of terms)}} + \underbrace{b^2}_{\text{(last term)}^2}$$

To square a binomial:

- square the first term
- square the last term
- double their product

A number example helps verify the pattern.

	$(10+4)^2$
Square the fir t term.	$10^2 + \_\_\_ + \_\_\_$
Square the last term.	$10^2 + \_\_\_ + 4^2$
Double their product.	$10^2 + 2 \cdot 10 \cdot 4 + 4^2$
Simplify.	$100 + 80 + 16$
Simplify.	196

To multiply $(10+4)^2$ usually you'd follow the Order of Operations.

$$(10+4)^2$$
$$(14)^2$$
$$196$$

The pattern works!

EXAMPLE 6.47

Multiply: $(x+5)^2$.

Solution

	$\left(\overset{a+b}{x+5}\right)^2$
Square the first term.	$\overset{a^2}{x^2} \overset{+}{+} \overset{2ab}{\_\_\_} \overset{+}{+} \overset{b^2}{\_\_\_}$
Square the last term.	$\overset{a^2}{x^2} \overset{+}{+} \overset{2ab}{\_\_\_} \overset{+}{+} \overset{b^2}{5^2}$
Double the product.	$\overset{a^2}{x^2} \overset{+}{+} \overset{2}{2} \overset{\cdot}{\cdot} \overset{a}{x} \overset{\cdot}{\cdot} \overset{b}{5} \overset{+}{+} \overset{b^2}{5^2}$
Simplify.	$x^2 + 10x + 25$

TRY IT : : 6.93 Multiply: $(x + 9)^2$.

TRY IT : : 6.94 Multiply: $(y + 11)^2$.

EXAMPLE 6.48

Multiply: $(y - 3)^2$.

Solution

	$\left(\overset{a-b}{y-3}\right)^2$
Square the first term.	$\overset{a^2}{y^2} \overset{-}{-} \overset{2ab}{\_\_\_} \overset{+}{+} \overset{b^2}{\_\_\_}$
Square the last term.	$\overset{a^2}{y^2} \overset{-}{-} \overset{2ab}{\_\_\_} \overset{+}{+} \overset{b^2}{3^2}$
Double the product.	$\overset{a^2}{y^2} \overset{-}{-} \overset{2}{2} \overset{\cdot}{\cdot} \overset{a}{y} \overset{\cdot}{\cdot} \overset{b}{3} \overset{+}{+} \overset{b^2}{3^2}$
Simplify.	$y^2 - 6y + 9$

TRY IT : : 6.95 Multiply: $(x - 9)^2$.

TRY IT : : 6.96 Multiply: $(p - 13)^2$.

EXAMPLE 6.49

Multiply: $(4x + 6)^2$.

Solution

	$\overset{a\ \ +\ \ b}{(4x + 6)}^2$
Use the pattern.	$\overset{a^2}{(4x)^2} + \overset{2\ \cdot\ a\ \cdot\ b}{2 \cdot 4x \cdot 6} + \overset{b^2}{6^2}$
Simplify.	$16x^2 + 48x + 36$

TRY IT : : 6.97 Multiply: $(6x+3)^2$.

TRY IT : : 6.98 Multiply: $(4x+9)^2$.

EXAMPLE 6.50

Multiply: $(2x-3y)^2$.

Solution

	$\overset{a\ \ -\ \ b}{(2x - 3y)}^2$
Use the pattern.	$\overset{a^2}{(2x)^2} - \overset{2\ \cdot\ a\ \cdot\ b}{2 \cdot 2x \cdot 3y} + \overset{b^2}{(3y)^2}$
Simplify.	$4x^2 - 12xy + 9y^2$

TRY IT : : 6.99 Multiply: $(2c-d)^2$.

TRY IT : : 6.100 Multiply: $(4x-5y)^2$.

EXAMPLE 6.51

Multiply: $\left(4u^3+1\right)^2$.

Solution

	$\overset{a\ \ +\ \ b}{(4u^3 + 1)}^2$
Use the pattern.	$\overset{a^2}{(4u^3)^2} + \overset{2\ \cdot\ a\ \cdot\ b}{2 \cdot 4u^3 \cdot 1} + \overset{b^2}{(1)^2}$
Simplify.	$16u^6 + 8u^3 + 1$

TRY IT : : 6.101 Multiply: $\left(2x^2+1\right)^2$.

TRY IT : : 6.102 Multiply: $\left(3y^3+2\right)^2$.

Multiply Conjugates Using the Product of Conjugates Pattern

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials. But before we get to it, we need to introduce some vocabulary.

What do you notice about these pairs of binomials?

$$(x-9)(x+9) \qquad (y-8)(y+8) \qquad (2x-5)(2x+5)$$

Look at the first term of each binomial in each pair.

$$(x-9)(x+9) \qquad (y-8)(y+8) \qquad (2x-5)(2x+5)$$

Notice the first terms are the same in each pair.

Look at the last terms of each binomial in each pair.

$$(x-9)(x+9) \qquad (y-8)(y+8) \qquad (2x-5)(2x+5)$$

Notice the last terms are the same in each pair.

Notice how each pair has one sum and one difference.

$$\left(\underset{\text{Difference}}{x-9}\right)\left(\underset{\text{Sum}}{x+9}\right) \qquad \left(\underset{\text{Difference}}{y-8}\right)\left(\underset{\text{Sum}}{y+8}\right) \qquad \left(\underset{\text{Difference}}{2x-5}\right)\left(\underset{\text{Sum}}{2x+5}\right)$$

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference has a special name. It is called a *conjugate pair* and is of the form $(a-b),\ (a+b)$.

Conjugate Pair

A **conjugate pair** is two binomials of the form

$$(a-b),\ (a+b).$$

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier.

Let's look for the pattern by using FOIL to multiply some conjugate pairs.

$$\begin{array}{ccc} (x-9)(x+9) & (y-8)(y+8) & (2x-5)(2x+5) \\ x^2+9x-9x-81 & y^2+8y-8y-64 & 4x^2+10x-10x-25 \\ x^2-81 & y^2-64 & 4x^2-25 \end{array}$$

$$\begin{array}{ccc} (x+9)(x-9) & (y-8)(y+8) & (2x-5)(2x+5) \\ x^2-9x+9x-81 & y^2+8y-8y-64 & 4x^2+10x-10x-25 \\ x^2-81 & y^2-64 & 4x^2-25 \end{array}$$

Each **first term** is the product of the first terms of the binomials, and since they are identical it is the square of the first term.

$$(a+b)(a-b)=a^2-\underline{\qquad}$$

To get the **fir t term, square the fir t term**.

The **last term** came from multiplying the last terms, the square of the last term.

$$(a+b)(a-b)=a^2-b^2$$

To get the **last term, square the last term**.

What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form $a^2 - b^2$. This is called a difference of squares.

This leads to the pattern:

Product of Conjugates Pattern

If a and b are real numbers,

$$(a-b)(a+b) = a^2 - b^2 \qquad \underbrace{(a-b)(a+b)}_{\text{conjugates}} = \underbrace{a^2 \overset{\text{difference}}{-} b^2}_{\text{squares}}$$

The product is called a difference of squares.

To multiply conjugates, square the first term, square the last term, and write the product as a difference of squares.

Let's test this pattern with a numerical example.

	$(10 - 2)(10 + 2)$
It is the product of conjugates, so the result will be the diffe ence of two squares.	$\_\_\_\_ - \_\_\_\_$
Square the fir t term.	$10^2 - \_\_\_\_$
Square the last term.	$10^2 - 2^2$
Simplify.	$100 - 4$
Simplify.	96
What do you get using the Order of Operations?	
	$(10 - 2)(10 + 2)$
	$(8)(12)$
	96

Notice, the result is the same!

EXAMPLE 6.52

Multiply: $(x - 8)(x + 8)$.

Solution

First, recognize this as a product of conjugates. The binomials have the same first terms, and the same last terms, and one binomial is a sum and the other is a difference.

It fits the pattern.	$\left(\overset{a\ -\ b}{x - 8}\right)\left(\overset{a\ +\ b}{x + 8}\right)$
Square the first term, x.	$\overset{a^2\ -\ b^2}{x^2 - \_\_}$
Square the last term, 8.	$\overset{a^2\ -\ b^2}{x^2 - 8^2}$
The product is a difference of squares.	$\overset{a^2\ -\ b^2}{x^2 - 64}$

TRY IT : : 6.103 Multiply: $(x - 5)(x + 5)$.

TRY IT : : 6.104 Multiply: $(w - 3)(w + 3)$.

EXAMPLE 6.53

Multiply: $(2x+5)(2x-5)$.

Solution

Are the binomials conjugates?

It is the product of conjugates.	$\overset{a\ +\ b}{(2x+5)}\overset{a\ -\ b}{(2x-5)}$
Square the first term, $2x$.	$\overset{a^2\ -\ b^2}{(2x)^2 - \_\_}$
Square the last term, 5.	$\overset{a^2\ -\ b^2}{(2x)^2 - 5^2}$
Simplify. The product is a difference of squares.	$\overset{a^2\ -\ b^2}{4x^2 - 25}$

TRY IT : : 6.105 Multiply: $(6x+5)(6x-5)$.

TRY IT : : 6.106 Multiply: $(2x+7)(2x-7)$.

The binomials in the next example may look backwards – the variable is in the second term. But the two binomials are still conjugates, so we use the same pattern to multiply them.

EXAMPLE 6.54

Find the product: $(3+5x)(3-5x)$.

Solution

It is the product of conjugates.	$\overset{a\ -\ b}{(3+5x)}\overset{a\ +\ b}{(3-5x)}$
Use the pattern.	$\overset{a^2\ -\ b^2}{3^2 - (5x)^2}$
Simplify.	$9 - 25x^2$

TRY IT : : 6.107 Multiply: $(7+4x)(7-4x)$.

TRY IT : : 6.108 Multiply: $(9-2y)(9+2y)$.

Now we'll multiply conjugates that have two variables.

EXAMPLE 6.55

Find the product: $(5m-9n)(5m+9n)$.

Solution

This fits the pattern.	$\left(\overset{a}{5m} \overset{-}{-} \overset{b}{9n}\right)\left(\overset{a}{5m} \overset{+}{+} \overset{b}{9n}\right)$
Use the pattern.	$\overset{a^2}{(5m)^2} \overset{-}{-} \overset{b^2}{(9n)^2}$
Simplify.	$25m^2 - 81n^2$

TRY IT : : 6.109 Find the product: $(4p - 7q)(4p + 7q)$.

TRY IT : : 6.110 Find the product: $(3x - y)(3x + y)$.

EXAMPLE 6.56

Find the product: $(cd - 8)(cd + 8)$.

Solution

This fits the pattern.	$\left(\overset{a}{cd} \overset{-}{-} \overset{b}{8}\right)\left(\overset{a}{cd} \overset{+}{+} \overset{b}{8}\right)$
Use the pattern.	$\overset{a^2}{(cd)^2} \overset{-}{-} \overset{b^2}{(8)^2}$
Simplify.	$c^2d^2 - 64$

TRY IT : : 6.111 Find the product: $(xy - 6)(xy + 6)$.

TRY IT : : 6.112 Find the product: $(ab - 9)(ab + 9)$.

EXAMPLE 6.57

Find the product: $\left(6u^2 - 11v^5\right)\left(6u^2 + 11v^5\right)$.

Solution

This fits the pattern.	$\left(\overset{a}{6u^2} \overset{-}{-} \overset{b}{11v^5}\right)\left(\overset{a}{6u^2} \overset{+}{+} \overset{b}{11v^5}\right)$
Use the pattern.	$\overset{a^2}{(6u^2)^2} \overset{-}{-} \overset{b^2}{(11v^5)^2}$
Simplify.	$36u^4 - 121v^{10}$

TRY IT : : 6.113 Find the product: $\left(3x^2 - 4y^3\right)\left(3x^2 + 4y^3\right)$.

TRY IT : : 6.114 Find the product: $\left(2m^2 - 5n^3\right)\left(2m^2 + 5n^3\right)$.

Recognize and Use the Appropriate Special Product Pattern

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.

Comparing the Special Product Patterns

Binomial Squares	**Product of Conjugates**
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)(a+b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	
- Squaring a binomial	- Multiplying conjugates
- Product is a **trinomial**	- Product is a **binomial**
- Inner and outer terms with FOIL are **the same.**	- Inner and outer terms with FOIL are **opposites.**
- Middle term is **double the product** of the terms.	- There is **no** middle term.

EXAMPLE 6.58

Choose the appropriate pattern and use it to find the product:

ⓐ $(2x-3)(2x+3)$ ⓑ $(5x-8)^2$ ⓒ $(6m+7)^2$ ⓓ $(5x-6)(6x+5)$

Solution

ⓐ $(2x-3)(2x+3)$ These are conjugates. They have the same first numbers, and the same last numbers, and one binomial is a sum and the other is a difference. It fits the Product of Conjugates pattern.

This fits the pattern.	$\left(\overset{a}{2x} \overset{-}{-} \overset{b}{3}\right)\left(\overset{a}{2x} \overset{+}{+} \overset{b}{3}\right)$
Use the pattern.	$\overset{a^2}{(2x)^2} \overset{-}{-} \overset{b^2}{3^2}$
Simplify.	$4x^2 - 9$

ⓑ $(8x-5)^2$ We are asked to square a binomial. It fits the **binomial squares** pattern.

	$\left(\overset{a}{8x} \overset{-}{-} \overset{b}{5}\right)^2$
Use the pattern.	$\overset{a^2}{(8x)^2} \overset{-}{-} \overset{2ab}{2 \cdot 8x \cdot 5} \overset{+}{+} \overset{b^2}{5^2}$
Simplify.	$64x^2 - 80x + 25$

ⓒ $(6m+7)^2$ Again, we will square a binomial so we use the **binomial squares** pattern.

	$\left(\overset{a}{6m} \overset{+}{+} \overset{b}{7}\right)^2$
Use the pattern.	$\overset{a^2}{(6m)^2} \overset{+}{+} \overset{2ab}{2 \cdot 6m \cdot 7} \overset{+}{+} \overset{b^2}{7^2}$
Simplify.	$36m^2 + 84m + 49$

ⓓ $(5x-6)(6x+5)$ This product does not fit the patterns, so we will use FOIL.

	$(5x-6)(6x+5)$
Use FOIL.	$30x^2 + 25x - 36x - 30$
Simplify.	$30x^2 - 11x - 30$

TRY IT : : 6.115 Choose the appropriate pattern and use it to find the product:

ⓐ $(9b-2)(2b+9)$ ⓑ $(9p-4)^2$ ⓒ $(7y+1)^2$ ⓓ $(4r-3)(4r+3)$

TRY IT : : 6.116 Choose the appropriate pattern and use it to find the product:

ⓐ $(6x+7)^2$ ⓑ $(3x-4)(3x+4)$ ⓒ $(2x-5)(5x-2)$ ⓓ $(6n-1)^2$

MEDIA : :

Access these online resources for additional instruction and practice with special products:

- **Special Products (https://openstax.org/l/25Specialprod)**

6.4 EXERCISES

Practice Makes Perfect

Square a Binomial Using the Binomial Squares Pattern

In the following exercises, square each binomial using the Binomial Squares Pattern.

302. $(w+4)^2$

303. $(q+12)^2$

304. $\left(y+\frac{1}{4}\right)^2$

305. $\left(x+\frac{2}{3}\right)^2$

306. $(b-7)^2$

307. $(y-6)^2$

308. $(m-15)^2$

309. $(p-13)^2$

310. $(3d+1)^2$

311. $(4a+10)^2$

312. $\left(2q+\frac{1}{3}\right)^2$

313. $\left(3z+\frac{1}{5}\right)^2$

314. $(3x-y)^2$

315. $(2y-3z)^2$

316. $\left(\frac{1}{5}x-\frac{1}{7}y\right)^2$

317. $\left(\frac{1}{8}x-\frac{1}{9}y\right)^2$

318. $(3x^2+2)^2$

319. $(5u^2+9)^2$

320. $(4y^3-2)^2$

321. $(8p^3-3)^2$

Multiply Conjugates Using the Product of Conjugates Pattern

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

322. $(m-7)(m+7)$

323. $(c-5)(c+5)$

324. $\left(x+\frac{3}{4}\right)\left(x-\frac{3}{4}\right)$

325. $\left(b+\frac{6}{7}\right)\left(b-\frac{6}{7}\right)$

326. $(5k+6)(5k-6)$

327. $(8j+4)(8j-4)$

328. $(11k+4)(11k-4)$

329. $(9c+5)(9c-5)$

330. $(11-b)(11+b)$

331. $(13-q)(13+q)$

332. $(5-3x)(5+3x)$

333. $(4-6y)(4+6y)$

334. $(9c-2d)(9c+2d)$

335. $(7w+10x)(7w-10x)$

336. $\left(m+\frac{2}{3}n\right)\left(m-\frac{2}{3}n\right)$

337. $\left(p+\frac{4}{5}q\right)\left(p-\frac{4}{5}q\right)$

338. $(ab-4)(ab+4)$

339. $(xy-9)(xy+9)$

340. $(uv-\frac{3}{5})(uv+\frac{3}{5})$

341. $(rs-\frac{2}{7})(rs+\frac{2}{7})$

342. $(2x^2-3y^4)(2x^2+3y^4)$

343. $(6m^3-4n^5)(6m^3+4n^5)$

344. $(12p^3-11q^2)(12p^3+11q^2)$

345. $(15m^2-8n^4)(15m^2+8n^4)$

Recognize and Use the Appropriate Special Product Pattern

In the following exercises, find each product.

346.

ⓐ $(p-3)(p+3)$

ⓑ $(t-9)^2$

ⓒ $(m+n)^2$

ⓓ $(2x+y)(x-2y)$

347.

ⓐ $(2r+12)^2$

ⓑ $(3p+8)(3p-8)$

ⓒ $(7a+b)(a-7b)$

ⓓ $(k-6)^2$

348.

ⓐ $(a^5-7b)^2$

ⓑ $(x^2+8y)(8x-y^2)$

ⓒ $(r^6+s^6)(r^6-s^6)$

ⓓ $(y^4+2z)^2$

349.

ⓐ $(x^5+y^5)(x^5-y^5)$

ⓑ $(m^3-8n)^2$

ⓒ $(9p+8q)^2$

ⓓ $(r^2-s^3)(r^3+s^2)$

Everyday Math

350. Mental math You can use the product of conjugates pattern to multiply numbers without a calculator. Say you need to multiply 47 times 53. Think of 47 as $50-3$ and 53 as $50+3$.

ⓐ Multiply $(50-3)(50+3)$ by using the product of conjugates pattern, $(a-b)(a+b)=a^2-b^2$.

ⓑ Multiply $47 \cdot 53$ without using a calculator.

ⓒ Which way is easier for you? Why?

351. Mental math You can use the binomial squares pattern to multiply numbers without a calculator. Say you need to square 65. Think of 65 as $60+5$.

ⓐ Multiply $(60+5)^2$ by using the binomial squares pattern, $(a+b)^2=a^2+2ab+b^2$.

ⓑ Square 65 without using a calculator.

ⓒ Which way is easier for you? Why?

Writing Exercises

352. How do you decide which pattern to use?

353. Why does $(a+b)^2$ result in a trinomial, but $(a-b)(a+b)$ result in a binomial?

354. Marta did the following work on her homework paper:

$$(3-y)^2$$
$$3^2-y^2$$
$$9-y^2$$

Explain what is wrong with Marta's work.

355. Use the order of operations to show that $(3+5)^2$ is 64, and then use that numerical example to explain why $(a+b)^2 \neq a^2+b^2$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
square a binomial using the binomial squares pattern.			
multiply conjugates using the product of conjugates pattern.			
recognize and use the appropriate special product pattern.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

6.5 Divide Monomials

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions using the Quotient Property for Exponents
- Simplify expressions with zero exponents
- Simplify expressions using the quotient to a Power Property
- Simplify expressions by applying several properties
- Divide monomials

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{8}{24}$.
   If you missed this problem, review Example 1.65.
2. Simplify: $\left(2m^3\right)^5$.
   If you missed this problem, review Example 6.23.
3. Simplify: $\frac{12x}{12y}$.
   If you missed this problem, review Example 1.67.

Simplify Expressions Using the Quotient Property for Exponents

Earlier in this chapter, we developed the properties of exponents for multiplication. We summarize these properties below.

Summary of Exponent Properties for Multiplication

If a and b are real numbers, and m and n are whole numbers, then

$$\begin{array}{lrcl} \textbf{Product Property} & a^m \cdot a^n & = & a^{m+n} \\ \textbf{Power Property} & (a^m)^n & = & a^{m \cdot n} \\ \textbf{Product to a Power} & (ab)^m & = & a^m b^m \end{array}$$

Now we will look at the exponent properties for division. A quick memory refresher may help before we get started. You have learned to simplify fractions by dividing out common factors from the numerator and denominator using the Equivalent Fractions Property. This property will also help you work with algebraic fractions—which are also quotients.

Equivalent Fractions Property

If $a, b,$ and c are whole numbers where $b \neq 0, c \neq 0$,

$$\text{then} \quad \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \quad \text{and} \quad \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

As before, we'll try to discover a property by looking at some examples.

Consider	$\frac{x^5}{x^2}$	and	$\frac{x^2}{x^3}$
What do they mean?	$\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$		$\frac{x \cdot x}{x \cdot x \cdot x}$
Use the Equivalent Fractions Property.	$\frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}}$		$\frac{\cancel{x} \cdot \cancel{x} \cdot 1}{\cancel{x} \cdot \cancel{x} \cdot x}$
Simplify.	x^3		$\frac{1}{x}$

Notice, in each case the bases were the same and we subtracted exponents.

When the larger exponent was in the numerator, we were left with factors in the numerator.

When the larger exponent was in the denominator, we were left with factors in the denominator—notice the numerator of 1.

We write:

$$\begin{array}{cc} \dfrac{x^5}{x^2} & \dfrac{x^2}{x^3} \\ x^{5-2} & \dfrac{1}{x^{3-2}} \\ x^3 & \dfrac{1}{x} \end{array}$$

This leads to the *Quotient Property for Exponents*.

Quotient Property for Exponents

If a is a real number, $a \neq 0$, and m and n are whole numbers, then

$$\frac{a^m}{a^n} = a^{m-n},\ m > n \ \text{ and } \ \frac{a^m}{a^n} = \frac{1}{a^{n-m}},\ n > m$$

A couple of examples with numbers may help to verify this property.

$$\begin{array}{rcl} \dfrac{3^4}{3^2} & = & 3^{4-2} \\ \dfrac{81}{9} & = & 3^2 \\ 9 & = & 9\ \checkmark \end{array} \qquad \begin{array}{rcl} \dfrac{5^2}{5^3} & = & \dfrac{1}{5^{3-2}} \\ \dfrac{25}{125} & = & \dfrac{1}{5^1} \\ \dfrac{1}{5} & = & \dfrac{1}{5}\ \checkmark \end{array}$$

EXAMPLE 6.59

Simplify: ⓐ $\dfrac{x^9}{x^7}$ ⓑ $\dfrac{3^{10}}{3^2}$.

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

ⓐ

Since 9 > 7, there are more factors of *x* in the numerator.	$\dfrac{x^9}{x^7}$
Use the Quotient Property, $\dfrac{a^m}{a^n} = a^{m-n}$.	x^{9-7}
Simplify.	x^2

ⓑ

Since 10 > 2, there are more factors of *x* in the numerator.	$\dfrac{3^{10}}{3^2}$
Use the Quotient Property, $\dfrac{a^m}{a^n} = a^{m-n}$.	3^{10-2}
Simplify.	3^8

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

TRY IT : : 6.117

Simplify: ⓐ $\dfrac{x^{15}}{x^{10}}$ ⓑ $\dfrac{6^{14}}{6^5}$.

TRY IT : : 6.118 Simplify: ⓐ $\frac{y^{43}}{y^{37}}$ ⓑ $\frac{10^{15}}{10^{7}}$.

EXAMPLE 6.60

Simplify: ⓐ $\frac{b^8}{b^{12}}$ ⓑ $\frac{7^3}{7^5}$.

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

ⓐ

Since 12 > 8, there are more factors of *b* in the denominator.	$\frac{b^8}{b^{12}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{b^{12-8}}$
Simplify.	$\frac{1}{b^4}$

ⓑ

Since 5 > 3, there are more factors of 3 in the denominator.	$\frac{7^3}{7^5}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{7^{5-3}}$
Simplify.	$\frac{1}{7^2}$
Simplify.	$\frac{1}{49}$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

TRY IT : : 6.119 Simplify: ⓐ $\frac{x^{18}}{x^{22}}$ ⓑ $\frac{12^{15}}{12^{30}}$.

TRY IT : : 6.120 Simplify: ⓐ $\frac{m^7}{m^{15}}$ ⓑ $\frac{9^8}{9^{19}}$.

Notice the difference in the two previous examples:

- If we start with more factors in the numerator, we will end up with factors in the numerator.
- If we start with more factors in the denominator, we will end up with factors in the denominator.

The first step in simplifying an expression using the Quotient Property for Exponents is to determine whether the exponent is larger in the numerator or the denominator.

EXAMPLE 6.61

Simplify: ⓐ $\frac{a^5}{a^9}$ ⓑ $\frac{x^{11}}{x^7}$.

Solution

ⓐ Is the exponent of a larger in the numerator or denominator? Since 9 > 5, there are more a's in the denominator and so we will end up with factors in the denominator.

	$\frac{a^5}{a^9}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{a^{9-5}}$
Simplify.	$\frac{1}{a^4}$

ⓑ Notice there are more factors of x in the numerator, since 11 > 7. So we will end up with factors in the numerator.

	$\frac{x^{11}}{x^7}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	x^{11-7}
Simplify.	x^4

TRY IT : : 6.121 Simplify: ⓐ $\frac{b^{19}}{b^{11}}$ ⓑ $\frac{z^5}{z^{11}}$.

TRY IT : : 6.122 Simplify: ⓐ $\frac{p^9}{p^{17}}$ ⓑ $\frac{w^{13}}{w^9}$.

Simplify Expressions with an Exponent of Zero

A special case of the Quotient Property is when the exponents of the numerator and denominator are equal, such as an expression like $\frac{a^m}{a^m}$. From your earlier work with fractions, you know that:

$$\frac{2}{2} = 1 \qquad \frac{17}{17} = 1 \qquad \frac{-43}{-43} = 1$$

In words, a number divided by itself is 1. So, $\frac{x}{x} = 1$, for any $x\ (x \neq 0)$, since any number divided by itself is 1.

The Quotient Property for Exponents shows us how to simplify $\frac{a^m}{a^n}$ when $m > n$ and when $n < m$ by subtracting exponents. What if $m = n$?

Consider $\frac{8}{8}$, which we know is 1.

	$\frac{8}{8}$	$= 1$
Write 8 as 2^3.	$\frac{2^3}{2^3}$	$= 1$
Subtract exponents.	2^{3-3}	$= 1$
Simplify.	2^0	$= 1$

Now we will simplify $\frac{a^m}{a^m}$ in two ways to lead us to the definition of the zero exponent. In general, for $a \neq 0$:

$$\frac{a^m}{a^m} \qquad \frac{a^m}{a^m}$$

$$a^{m-m} \qquad \frac{\overbrace{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}}^{m \text{ factors}}}{\underbrace{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}}_{m \text{ factors}}}$$

$$a^0 \qquad 1$$

We see $\frac{a^m}{a^m}$ simplifies to a^0 and to 1. So $a^0 = 1$.

Zero Exponent

If a is a non-zero number, then $a^0 = 1$.

Any nonzero number raised to the zero power is 1.

In this text, we assume any variable that we raise to the zero power is not zero.

EXAMPLE 6.62

Simplify: ⓐ 9^0 ⓑ n^0.

Solution

The definition says any non-zero number raised to the zero power is 1.

ⓐ

	9^0
Use the definition of he zero exponent.	1

ⓑ

	n^0
Use the definition of he zero exponent.	1

TRY IT : : 6.123 Simplify: ⓐ 15^0 ⓑ m^0.

TRY IT : : 6.124 Simplify: ⓐ k^0 ⓑ 29^0.

Now that we have defined the zero exponent, we can expand all the Properties of Exponents to include whole number exponents.

What about raising an expression to the zero power? Let's look at $(2x)^0$. We can use the product to a power rule to rewrite this expression.

	$(2x)^0$
Use the product to a power rule.	$2^0 x^0$
Use the zero exponent property.	$1 \cdot 1$
Simplify.	1

This tells us that any nonzero expression raised to the zero power is one.

EXAMPLE 6.63

Simplify: ⓐ $(5b)^0$ ⓑ $\left(-4a^2 b\right)^0$.

Solution

ⓐ

	$(5b)^0$
Use the definition of he zero exponent.	1

ⓑ

	$\left(-4a^2b\right)^0$
Use the definition of he zero exponent.	1

TRY IT : : 6.125 Simplify: ⓐ $(11z)^0$ ⓑ $\left(-11pq^3\right)^0$.

TRY IT : : 6.126 Simplify: ⓐ $(-6d)^0$ ⓑ $\left(-8m^2n^3\right)^0$.

Simplify Expressions Using the Quotient to a Power Property

Now we will look at an example that will lead us to the Quotient to a Power Property.

	$\left(\frac{x}{y}\right)^3$
This means:	$\frac{x}{y}\cdot\frac{x}{y}\cdot\frac{x}{y}$
Multiply the fractions.	$\frac{x\cdot x\cdot x}{y\cdot y\cdot y}$
Write with exponents.	$\frac{x^3}{y^3}$

Notice that the exponent applies to both the numerator and the denominator.

We see that $\left(\frac{x}{y}\right)^3$ is $\frac{x^3}{y^3}$.

We write:	$\left(\frac{x}{y}\right)^3$
	$\frac{x^3}{y^3}$

This leads to the *Quotient to a Power Property for Exponents*.

Quotient to a Power Property for Exponents

If a and b are real numbers, $b \neq 0$, and m is a counting number, then

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

To raise a fraction to a power, raise the numerator and denominator to that power.

An example with numbers may help you understand this property:

$$\begin{aligned}\left(\frac{2}{3}\right)^3 &= \frac{2^3}{3^3}\\ \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} &= \frac{8}{27}\\ \frac{8}{27} &= \frac{8}{27}\ \checkmark\end{aligned}$$

EXAMPLE 6.64

Simplify: ⓐ $\left(\frac{3}{7}\right)^2$ ⓑ $\left(\frac{b}{3}\right)^4$ ⓒ $\left(\frac{k}{j}\right)^3$.

Solution

ⓐ

	$\left(\frac{3}{7}\right)^2$
Use the Quotient Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{3^2}{7^2}$
Simplify.	$\frac{9}{49}$

ⓑ

	$\left(\frac{b}{3}\right)^4$
Use the Quotient Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{b^4}{3^4}$
Simplify.	$\frac{b^4}{81}$

ⓒ

	$\left(\frac{k}{j}\right)^3$
Raise the numerator and denominator to the third power.	$\frac{k^3}{j^3}$

TRY IT : : 6.127 Simplify: ⓐ $\left(\frac{5}{8}\right)^2$ ⓑ $\left(\frac{p}{10}\right)^4$ ⓒ $\left(\frac{m}{n}\right)^7$.

TRY IT : : 6.128 Simplify: ⓐ $\left(\frac{1}{3}\right)^3$ ⓑ $\left(\frac{-2}{q}\right)^3$ ⓒ $\left(\frac{w}{x}\right)^4$.

Simplify Expressions by Applying Several Properties

We'll now summarize all the properties of exponents so they are all together to refer to as we simplify expressions using several properties. Notice that they are now defined for whole number exponents.

Summary of Exponent Properties

If a and b are real numbers, and m and n are whole numbers, then

Product Property	$a^m \cdot a^n$	$=$	a^{m+n}
Power Property	$(a^m)^n$	$=$	$a^{m \cdot n}$
Product to a Power	$(ab)^m$	$=$	$a^m b^m$
Quotient Property	$\frac{a^m}{b^m}$	$=$	$a^{m-n},\ a \neq 0,\ m > n$
	$\frac{a^m}{a^n}$	$=$	$\frac{1}{a^{n-m}},\ a \neq 0,\ n > m$
Zero Exponent Definitio	a^o	$=$	$1,\ a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m$	$=$	$\frac{a^m}{b^m},\ b \neq 0$

EXAMPLE 6.65

Simplify: $\frac{\left(y^4\right)^2}{y^6}$.

Solution

	$\frac{\left(y^4\right)^2}{y^6}$
Multiply the exponents in the numerator.	$\frac{y^8}{y^6}$
Subtract the exponents.	y^2

TRY IT : : 6.129

Simplify: $\frac{\left(m^5\right)^4}{m^7}$.

TRY IT : : 6.130

Simplify: $\frac{\left(k^2\right)^6}{k^7}$.

EXAMPLE 6.66

Simplify: $\frac{b^{12}}{\left(b^2\right)^6}$.

Solution

	$\frac{b^{12}}{\left(b^2\right)^6}$
Multiply the exponents in the denominator.	$\frac{b^{12}}{b^{12}}$
Subtract the exponents.	b^0
Simplify.	1

Notice that after we simplified the denominator in the first step, the numerator and the denominator were equal. So the final value is equal to 1.

TRY IT : : 6.131 Simplify: $\dfrac{n^{12}}{\left(n^3\right)^4}$.

TRY IT : : 6.132 Simplify: $\dfrac{x^{15}}{\left(x^3\right)^5}$.

EXAMPLE 6.67

Simplify: $\left(\dfrac{y^9}{y^4}\right)^2$.

Solution

	$\left(\dfrac{y^9}{y^4}\right)^2$
Remember parentheses come before exponents. Notice the bases are the same, so we can simplify inside the parentheses. Subtract the exponents.	$\left(y^5\right)^2$
Multiply the exponents.	y^{10}

TRY IT : : 6.133 Simplify: $\left(\dfrac{r^5}{r^3}\right)^4$.

TRY IT : : 6.134 Simplify: $\left(\dfrac{v^6}{v^4}\right)^3$.

EXAMPLE 6.68

Simplify: $\left(\dfrac{j^2}{k^3}\right)^4$.

Solution

Here we cannot simplify inside the parentheses first, since the bases are not the same.

	$\left(\dfrac{j^2}{k^3}\right)^4$
Raise the numerator and denominator to the third power using the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\dfrac{\left(j^2\right)^4}{\left(k^3\right)^4}$
Use the Power Property and simplify.	$\dfrac{j^8}{k^{12}}$

TRY IT : : 6.135 Simplify: $\left(\dfrac{a^3}{b^2}\right)^4$.

TRY IT : : 6.136

Simplify: $\left(\dfrac{q^7}{r^5}\right)^3$.

EXAMPLE 6.69

Simplify: $\left(\dfrac{2m^2}{5n}\right)^4$.

Solution

	$\left(\dfrac{2m^2}{5n}\right)^4$
Raise the numerator and denominator to the fourth power, using the Quotient to a Power Property, $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.	$\dfrac{(2m^2)^4}{(5n)^4}$
Raise each factor to the fourth power.	$\dfrac{2^4(m^2)^4}{5^4 n^4}$
Use the Power Property and simplify.	$\dfrac{16m^8}{625n^4}$

TRY IT : : 6.137

Simplify: $\left(\dfrac{7x^3}{9y}\right)^2$.

TRY IT : : 6.138

Simplify: $\left(\dfrac{3x^4}{7y}\right)^2$.

EXAMPLE 6.70

Simplify: $\dfrac{\left(x^3\right)^4\left(x^2\right)^5}{\left(x^6\right)^5}$.

Solution

	$\dfrac{\left(x^3\right)^4\left(x^2\right)^5}{\left(x^6\right)^5}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$\dfrac{\left(x^{12}\right)\left(x^{10}\right)}{\left(x^{30}\right)}$
Add the exponents in the numerator.	$\dfrac{x^{22}}{x^{30}}$
Use the Quotient Property, $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$.	$\dfrac{1}{x^8}$

TRY IT : : 6.139

Simplify: $\dfrac{\left(a^2\right)^3\left(a^2\right)^4}{\left(a^4\right)^5}$.

TRY IT : : 6.140

Simplify: $\dfrac{\left(p^3\right)^4\left(p^5\right)^3}{\left(p^7\right)^6}$.

EXAMPLE 6.71

Simplify: $\dfrac{\left(10p^3\right)^2}{(5p)^3\left(2p^5\right)^4}$.

Solution

	$\dfrac{\left(10p^3\right)^2}{(5p)^3\left(2p^5\right)^4}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$\dfrac{(10)^2\left(p^3\right)^2}{(5)^3(p)^3(2)^4\left(p^5\right)^4}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$\dfrac{100p^6}{125p^3 \cdot 16p^{20}}$
Add the exponents in the denominator.	$\dfrac{100p^6}{125 \cdot 16p^{23}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\dfrac{100}{125 \cdot 16p^{17}}$
Simplify.	$\dfrac{1}{20p^{17}}$

TRY IT : : 6.141

Simplify: $\dfrac{\left(3r^3\right)^2\left(r^3\right)^7}{\left(r^3\right)^3}$.

TRY IT : : 6.142

Simplify: $\dfrac{\left(2x^4\right)^5}{\left(4x^3\right)^2\left(x^3\right)^5}$.

Divide Monomials

You have now been introduced to all the properties of exponents and used them to simplify expressions. Next, you'll see how to use these properties to divide monomials. Later, you'll use them to divide polynomials.

EXAMPLE 6.72

Find the quotient: $56x^7 \div 8x^3$.

Solution

	$56x^7 \div 8x^3$
Rewrite as a fraction.	$\frac{56x^7}{8x^3}$
Use fraction multiplication.	$\frac{56}{8} \cdot \frac{x^7}{x^3}$
Simplify and use the Quotient Property.	$7x^4$

TRY IT : : 6.143 Find the quotient: $42y^9 \div 6y^3$.

TRY IT : : 6.144 Find the quotient: $48z^8 \div 8z^2$.

EXAMPLE 6.73

Find the quotient: $\frac{45a^2b^3}{-5ab^5}$.

Solution

When we divide monomials with more than one variable, we write one fraction for each variable.

	$\frac{45a^2b^3}{-5ab^5}$
Use fraction multiplication.	$\frac{45}{-5} \cdot \frac{a^2}{a} \cdot \frac{b^3}{b^5}$
Simplify and use the Quotient Property.	$-9 \cdot a \cdot \frac{1}{b^2}$
Multiply.	$-\frac{9a}{b^2}$

TRY IT : : 6.145 Find the quotient: $\frac{-72a^7b^3}{8a^{12}b^4}$.

TRY IT : : 6.146 Find the quotient: $\frac{-63c^8d^3}{7c^{12}d^2}$.

EXAMPLE 6.74

Find the quotient: $\frac{24a^5b^3}{48ab^4}$.

Solution

	$\frac{24a^5b^3}{48ab^4}$
Use fraction multiplication.	$\frac{24}{48} \cdot \frac{a^5}{a} \cdot \frac{b^3}{b^4}$
Simplify and use the Quotient Property.	$\frac{1}{2} \cdot a^4 \cdot \frac{1}{b}$
Multiply.	$\frac{a^4}{2b}$

TRY IT : : 6.147 Find the quotient: $\frac{16a^7 b^6}{24ab^8}$.

TRY IT : : 6.148 Find the quotient: $\frac{27p^4 q^7}{-45p^{12} q}$.

Once you become familiar with the process and have practiced it step by step several times, you may be able to simplify a fraction in one step.

EXAMPLE 6.75

Find the quotient: $\frac{14x^7 y^{12}}{21x^{11} y^6}$.

Solution

Be very careful to simplify $\frac{14}{21}$ by dividing out a common factor, and to simplify the variables by subtracting their exponents.

	$\frac{14x^7 y^{12}}{21x^{11} y^6}$
Simplify and use the Quotient Property.	$\frac{2y^6}{3x^4}$

TRY IT : : 6.149 Find the quotient: $\frac{28x^5 y^{14}}{49x^9 y^{12}}$.

TRY IT : : 6.150 Find the quotient: $\frac{30m^5 n^{11}}{48m^{10} n^{14}}$.

In all examples so far, there was no work to do in the numerator or denominator before simplifying the fraction. In the next example, we'll first find the product of two monomials in the numerator before we simplify the fraction. This follows the order of operations. Remember, a fraction bar is a grouping symbol.

EXAMPLE 6.76

Find the quotient: $\frac{(6x^2 y^3)(5x^3 y^2)}{(3x^4 y^5)}$.

Solution

	$\frac{(6x^2 y^3)(5x^3 y^2)}{(3x^4 y^5)}$
Simplify the numerator.	$\frac{30x^5 y^5}{3x^4 y^5}$
Simplify.	$10x$

TRY IT : : 6.151 Find the quotient: $\frac{(6a^4 b^5)(4a^2 b^5)}{12a^5 b^8}$.

TRY IT : : 6.152

Find the quotient: $\frac{\left(-12x^6 y^9\right)\left(-4x^5 y^8\right)}{-12x^{10} y^{12}}$.

MEDIA : :

Access these online resources for additional instruction and practice with dividing monomials:

- **Rational Expressions (https://openstax.org/l/25RationalExp)**
- **Dividing Monomials (https://openstax.org/l/25DivideMono)**
- **Dividing Monomials 2 (https://openstax.org/l/25DivideMono2)**

6.5 EXERCISES

Practice Makes Perfect

Simplify Expressions Using the Quotient Property for Exponents

In the following exercises, simplify.

356. ⓐ $\frac{x^{18}}{x^3}$ ⓑ $\frac{5^{12}}{5^3}$

357. ⓐ $\frac{y^{20}}{y^{10}}$ ⓑ $\frac{7^{16}}{7^2}$

358. ⓐ $\frac{p^{21}}{p^7}$ ⓑ $\frac{4^{16}}{4^4}$

359. ⓐ $\frac{u^{24}}{u^3}$ ⓑ $\frac{9^{15}}{9^5}$

360. ⓐ $\frac{q^{18}}{q^{36}}$ ⓑ $\frac{10^2}{10^3}$

361. ⓐ $\frac{t^{10}}{t^{40}}$ ⓑ $\frac{8^3}{8^5}$

362. ⓐ $\frac{b}{b^9}$ ⓑ $\frac{4}{4^6}$

363. ⓐ $\frac{x}{x^7}$ ⓑ $\frac{10}{10^3}$

Simplify Expressions with Zero Exponents

In the following exercises, simplify.

364.
ⓐ 20^0
ⓑ b^0

365.
ⓐ 13^0
ⓑ k^0

366.
ⓐ -27^0
ⓑ $-\left(27^0\right)$

367.
ⓐ -15^0
ⓑ $-\left(15^0\right)$

368.
ⓐ $(25x)^0$
ⓑ $25x^0$

369.
ⓐ $(6y)^0$
ⓑ $6y^0$

370.
ⓐ $(12x)^0$
ⓑ $\left(-56p^4q^3\right)^0$

371.
ⓐ $7y^0\ (17y)^0$
ⓑ $\left(-93c^7d^{15}\right)^0$

372.
ⓐ $12n^0 - 18m^0$
ⓑ $(12n)^0 - (18m)^0$

373.
ⓐ $15r^0 - 22s^0$
ⓑ $(15r)^0 - (22s)^0$

Simplify Expressions Using the Quotient to a Power Property

In the following exercises, simplify.

374. ⓐ $\left(\frac{3}{4}\right)^3$ ⓑ $\left(\frac{p}{2}\right)^5$ ⓒ $\left(\frac{x}{y}\right)^6$

375. ⓐ $\left(\frac{2}{5}\right)^2$ ⓑ $\left(\frac{x}{3}\right)^4$ ⓒ $\left(\frac{a}{b}\right)^5$

376. ⓐ $\left(\frac{a}{3b}\right)^4$ ⓑ $\left(\frac{5}{4m}\right)^2$

377. ⓐ $\left(\frac{x}{2y}\right)^3$ ⓑ $\left(\frac{10}{3q}\right)^4$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify.

378. $\dfrac{\left(a^2\right)^3}{a^4}$

379. $\dfrac{\left(p^3\right)^4}{p^5}$

380. $\dfrac{\left(y^3\right)^4}{y^{10}}$

381. $\dfrac{\left(x^4\right)^5}{x^{15}}$

382. $\dfrac{u^6}{\left(u^3\right)^2}$

383. $\dfrac{v^{20}}{\left(v^4\right)^5}$

384. $\dfrac{m^{12}}{\left(m^8\right)^3}$

385. $\dfrac{n^8}{\left(n^6\right)^4}$

386. $\left(\dfrac{p^9}{p^3}\right)^5$

387. $\left(\dfrac{q^8}{q^2}\right)^3$

388. $\left(\dfrac{r^2}{r^6}\right)^3$

389. $\left(\dfrac{m^4}{m^7}\right)^4$

390. $\left(\dfrac{p}{r^{11}}\right)^2$

391. $\left(\dfrac{a}{b^6}\right)^3$

392. $\left(\dfrac{w^5}{x^3}\right)^8$

393. $\left(\dfrac{y^4}{z^{10}}\right)^5$

394. $\left(\dfrac{2j^3}{3k}\right)^4$

395. $\left(\dfrac{3m^5}{5n}\right)^3$

396. $\left(\dfrac{3c^2}{4d^6}\right)^3$

397. $\left(\dfrac{5u^7}{2v^3}\right)^4$

398. $\left(\dfrac{k^2k^8}{k^3}\right)^2$

399. $\left(\dfrac{j^2j^5}{j^4}\right)^3$

400. $\dfrac{\left(t^2\right)^5\left(t^4\right)^2}{\left(t^3\right)^7}$

401. $\dfrac{\left(q^3\right)^6\left(q^2\right)^3}{\left(q^4\right)^8}$

402. $\dfrac{\left(-2p^2\right)^4\left(3p^4\right)^2}{\left(-6p^3\right)^2}$

403. $\dfrac{\left(-2k^3\right)^2\left(6k^2\right)^4}{\left(9k^4\right)^2}$

404. $\dfrac{\left(-4m^3\right)^2\left(5m^4\right)^3}{\left(-10m^6\right)^3}$

405. $\dfrac{\left(-10n^2\right)^3\left(4n^5\right)^2}{\left(2n^8\right)^2}$

Divide Monomials

In the following exercises, divide the monomials.

406. $56b^8 \div 7b^2$

407. $63v^{10} \div 9v^2$

408. $-88y^{15} \div 8y^3$

409. $-72u^{12} \div 12u^4$

410. $\dfrac{45a^6b^8}{-15a^{10}b^2}$

411. $\dfrac{54x^9y^3}{-18x^6y^{15}}$

412. $\frac{15r^4 s^9}{18r^9 s^2}$

413. $\frac{20m^8 n^4}{30m^5 n^9}$

414. $\frac{18a^4 b^8}{-27a^9 b^5}$

415. $\frac{45x^5 y^9}{-60x^8 y^6}$

416. $\frac{64q^{11} r^9 s^3}{48q^6 r^8 s^5}$

417. $\frac{65a^{10} b^8 c^5}{42a^7 b^6 c^8}$

418. $\frac{(10m^5 n^4)(5m^3 n^6)}{25m^7 n^5}$

419. $\frac{(-18p^4 q^7)(-6p^3 q^8)}{-36p^{12} q^{10}}$

420. $\frac{(6a^4 b^3)(4ab^5)}{(12a^2 b)(a^3 b)}$

421. $\frac{(4u^2 v^5)(15u^3 v)}{(12u^3 v)(u^4 v)}$

Mixed Practice

422.
ⓐ $24a^5 + 2a^5$
ⓑ $24a^5 - 2a^5$
ⓒ $24a^5 \cdot 2a^5$
ⓓ $24a^5 \div 2a^5$

423.
ⓐ $15n^{10} + 3n^{10}$
ⓑ $15n^{10} - 3n^{10}$
ⓒ $15n^{10} \cdot 3n^{10}$
ⓓ $15n^{10} \div 3n^{10}$

424.
ⓐ $p^4 \cdot p^6$
ⓑ $(p^4)^6$

425.
ⓐ $q^5 \cdot q^3$
ⓑ $(q^5)^3$

426.
ⓐ $\frac{y^3}{y}$
ⓑ $\frac{y}{y^3}$

427.
ⓐ $\frac{z^6}{z^5}$
ⓑ $\frac{z^5}{z^6}$

428. $(8x^5)(9x) \div 6x^3$

429. $(4y)(12y^7) \div 8y^2$

430. $\frac{27a^7}{3a^3} + \frac{54a^9}{9a^5}$

431. $\frac{32c^{11}}{4c^5} + \frac{42c^9}{6c^3}$

432. $\frac{32y^5}{8y^2} - \frac{60y^{10}}{5y^7}$

433. $\frac{48x^6}{6x^4} - \frac{35x^9}{7x^7}$

434. $\frac{63r^6 s^3}{9r^4 s^2} - \frac{72r^2 s^2}{6s}$

435. $\frac{56y^4 z^5}{7y^3 z^3} - \frac{45y^2 z^2}{5y}$

Everyday Math

436. Memory One megabyte is approximately 10^6 bytes. One gigabyte is approximately 10^9 bytes. How many megabytes are in one gigabyte?

437. Memory One gigabyte is approximately 10^9 bytes. One terabyte is approximately 10^{12} bytes. How many gigabytes are in one terabyte?

Writing Exercises

438. Jennifer thinks the quotient $\frac{a^{24}}{a^6}$ simplifies to a^4. What is wrong with her reasoning?

439. Maurice simplifies the quotient $\frac{d^7}{d}$ by writing $\frac{\cancel{d}^7}{\cancel{d}} = 7$. What is wrong with his reasoning?

440. When Drake simplified -3^0 and $(-3)^0$ he got the same answer. Explain how using the Order of Operations correctly gives different answers.

441. Robert thinks x^0 simplifies to 0. What would you say to convince Robert he is wrong?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify expressions using the Quotient Property for Exponents.			
simplify expressions with zero exponents.			
simplify expressions using the Quotient to a Power Property.			
simplify expressions by applying several properties.			
divide monomials.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

6.6 Divide Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Divide a polynomial by a monomial
- Divide a polynomial by a binomial

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $\frac{3}{d}+\frac{x}{d}$.
   If you missed this problem, review Example 1.77.
2. Simplify: $\frac{30xy^3}{5xy}$.
   If you missed this problem, review Example 6.72.
3. Combine like terms: $8a^2+12a+1+3a^2-5a+4$.
   If you missed this problem, review Example 1.24.

Divide a Polynomial by a Monomial

In the last section, you learned how to divide a monomial by a monomial. As you continue to build up your knowledge of polynomials the next procedure is to divide a polynomial of two or more terms by a monomial.

The method we'll use to divide a polynomial by a monomial is based on the properties of fraction addition. So we'll start with an example to review fraction addition.

The sum,	$\frac{y}{5}+\frac{2}{5},$
simplifies o	$\frac{y+2}{5}.$

Now we will do this in reverse to split a single fraction into separate fractions.

We'll state the fraction addition property here just as you learned it and in reverse.

Fraction Addition

If $a, b,$ and c are numbers where $c \neq 0$, then

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \quad \text{and} \quad \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$

We use the form on the left to add fractions and we use the form on the right to divide a polynomial by a monomial.

For example,	$\frac{y+2}{5}$
can be written	$\frac{y}{5}+\frac{2}{5}.$

We use this form of fraction addition to divide polynomials by monomials.

Division of a Polynomial by a Monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

EXAMPLE 6.77

Find the quotient: $\frac{7y^2+21}{7}$.

Solution

	$\dfrac{7y^2+21}{7}$
Divide each term of the numerator by the denominator.	$\dfrac{7y^2}{7}+\dfrac{21}{7}$
Simplify each fraction.	y^2+3

TRY IT : : 6.153 Find the quotient: $\dfrac{8z^2+24}{4}$.

TRY IT : : 6.154 Find the quotient: $\dfrac{18z^2-27}{9}$.

Remember that division can be represented as a fraction. When you are asked to divide a polynomial by a monomial and it is not already in fraction form, write a fraction with the polynomial in the numerator and the monomial in the denominator.

EXAMPLE 6.78

Find the quotient: $\left(18x^3-36x^2\right)\div 6x$.

Solution

	$\left(18x^3-36x^2\right)\div 6x$
Rewrite as a fraction.	$\dfrac{18x^3-36x^2}{6x}$
Divide each term of the numerator by the denominator.	$\dfrac{18x^3}{6x}-\dfrac{36x^2}{6x}$
Simplify.	$3x^2-6x$

TRY IT : : 6.155 Find the quotient: $\left(27b^3-33b^2\right)\div 3b$.

TRY IT : : 6.156 Find the quotient: $\left(25y^3-55y^2\right)\div 5y$.

When we divide by a negative, we must be extra careful with the signs.

EXAMPLE 6.79

Find the quotient: $\dfrac{12d^2-16d}{-4}$.

Solution

	$\dfrac{12d^2-16d}{-4}$
Divide each term of the numerator by the denominator.	$\dfrac{12d^2}{-4}-\dfrac{16d}{-4}$
Simplify. Remember, subtracting a negative is like adding a positive!	$-3d^2+4d$

TRY IT : : 6.157 Find the quotient: $\dfrac{25y^2-15y}{-5}$.

TRY IT : : 6.158 Find the quotient: $\frac{42b^2 - 18b}{-6}$.

EXAMPLE 6.80

Find the quotient: $\frac{105y^5 + 75y^3}{5y^2}$.

Solution

	$\frac{105y^5 + 75y^3}{5y^2}$
Separate the terms.	$\frac{105y^5}{5y^2} + \frac{75y^3}{5y^2}$
Simplify.	$21y^3 + 15y$

TRY IT : : 6.159 Find the quotient: $\frac{60d^7 + 24d^5}{4d^3}$.

TRY IT : : 6.160 Find the quotient: $\frac{216p^7 - 48p^5}{6p^3}$.

EXAMPLE 6.81

Find the quotient: $\left(15x^3y - 35xy^2\right) \div (-5xy)$.

Solution

	$\left(15x^3y - 35xy^2\right) \div (-5xy)$
Rewrite as a fraction.	$\frac{15x^3y - 35xy^2}{-5xy}$
Separate the terms. Be careful with the signs!	$\frac{15x^3y}{-5xy} - \frac{35xy^2}{-5xy}$
Simplify.	$-3x^2 + 7y$

TRY IT : : 6.161 Find the quotient: $\left(32a^2b - 16ab^2\right) \div (-8ab)$.

TRY IT : : 6.162 Find the quotient: $\left(-48a^8b^4 - 36a^6b^5\right) \div \left(-6a^3b^3\right)$.

EXAMPLE 6.82

Find the quotient: $\frac{36x^3y^2 + 27x^2y^2 - 9x^2y^3}{9x^2y}$.

Solution

	$\dfrac{36x^3y^2 + 27x^2y^2 - 9x^2y^3}{9x^2y}$
Separate the terms.	$\dfrac{36x^3y^2}{9x^2y} + \dfrac{27x^2y^2}{9x^2y} - \dfrac{9x^2y^3}{9x^2y}$
Simplify.	$4xy + 3y - y^2$

TRY IT : : 6.163 Find the quotient: $\dfrac{40x^3y^2 + 24x^2y^2 - 16x^2y^3}{8x^2y}$.

TRY IT : : 6.164 Find the quotient: $\dfrac{35a^4b^2 + 14a^4b^3 - 42a^2b^4}{7a^2b^2}$.

EXAMPLE 6.83

Find the quotient: $\dfrac{10x^2 + 5x - 20}{5x}$.

Solution

	$\dfrac{10x^2 + 5x - 20}{5x}$
Separate the terms.	$\dfrac{10x^2}{5x} + \dfrac{5x}{5x} - \dfrac{20}{5x}$
Simplify.	$2x + 1 + \dfrac{4}{x}$

TRY IT : : 6.165 Find the quotient: $\dfrac{18c^2 + 6c - 9}{6c}$.

TRY IT : : 6.166 Find the quotient: $\dfrac{10d^2 - 5d - 2}{5d}$.

Divide a Polynomial by a Binomial

To divide a polynomial by a binomial, we follow a procedure very similar to long division of numbers. So let’s look carefully the steps we take when we divide a 3-digit number, 875, by a 2-digit number, 25.

We write the long division	$25\overline{)875}$
We divide the first two digits, 87, by 25.	$\begin{array}{r} 3 \\ 25\overline{)875} \end{array}$
We multiply 3 times 25 and write the product under the 87.	$\begin{array}{r} 3 \\ 25\overline{)875} \\ \underline{75} \end{array}$
Now we subtract 75 from 87.	$\begin{array}{r} 3 \\ 25\overline{)875} \\ \underline{-75} \\ 12 \end{array}$
Then we bring down the third digit of the dividend, 5.	$\begin{array}{r} 3 \\ 25\overline{)875} \\ \underline{-75} \\ 125 \end{array}$
Repeat the process, dividing 25 into 125.	$\begin{array}{r} 35 \\ 25\overline{)875} \\ \underline{-75} \\ 125 \\ \underline{-125} \end{array}$

We check division by multiplying the quotient by the divisor.

If we did the division correctly, the product should equal the dividend.

$$\begin{array}{c} 35 \cdot 25 \\ 875\,\checkmark \end{array}$$

Now we will divide a trinomial by a binomial. As you read through the example, notice how similar the steps are to the numerical example above.

EXAMPLE 6.84

Find the quotient: $\left(x^2 + 9x + 20\right) \div (x + 5)$.

Solution

	$(x^2 + 9x + 20) \div (x + 5)$
Write it as a long division problem.	
Be sure the dividend is in standard form.	$x + 5\overline{)\,x^2 + 9x + 20}$
Divide x^2 by x. It may help to ask yourself, "What do I need to multiply x by to get x^2?"	
Put the answer, x, in the quotient over the x term.	$\begin{array}{r} x \\ x + 5\overline{)\,x^2 + 9x + 20} \end{array}$

Multiply x times $x + 5$. Line up the like terms under the dividend.	$\begin{array}{r} x \\ x+5\overline{)\,x^2+9x+20} \\ \underline{x^2+5x} \end{array}$
Subtract $x^2 + 5x$ from $x^2 + 9x$.	
You may find it easier to change the signs and then add. Then bring down the last term, 20.	$\begin{array}{r} x \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \end{array}$
Divide $4x$ by x. It may help to ask yourself, "What do I need to multiply x by to get $4x$?"	
Put the answer, 4, in the quotient over the constant term.	$\begin{array}{r} x+4 \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \end{array}$
Multiply 4 times $x + 5$.	$\begin{array}{r} x+4 \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \\ \underline{4x+20} \end{array}$
Subtract $4x + 20$ from $4x + 20$.	$\begin{array}{r} x+4 \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \\ \underline{-4x+(-20)} \\ 0 \end{array}$
Check:	
Multiply the quotient by the divisor.	
$(x + 4)(x + 5)$	
You should get the dividend.	
$x^2 + 9x + 20$✓	

TRY IT : : 6.167 Find the quotient: $\left(y^2 + 10y + 21\right) \div (y + 3)$.

TRY IT : : 6.168 Find the quotient: $\left(m^2 + 9m + 20\right) \div (m + 4)$.

When the divisor has subtraction sign, we must be extra careful when we multiply the partial quotient and then subtract. It may be safer to show that we change the signs and then add.

EXAMPLE 6.85

Find the quotient: $\left(2x^2 - 5x - 3\right) \div (x - 3)$.

Solution

	$(2x^2-5x-3)\div(x-3)$
Write it as a long division problem.	
Be sure the dividend is in standard form.	$x-3\overline{)2x^2-5x-3}$
Divide $2x^2$ by x. Put the answer, $2x$, in the quotient over the x term.	$\begin{array}{r} 2x \\ x-3\overline{)2x^2-5x-3} \end{array}$
Multiply $2x$ times $x-3$. Line up the like terms under the dividend.	$\begin{array}{r} 2x \\ x-3\overline{)2x^2-5x-3} \\ \underline{2x^2-6x} \end{array}$
Subtract $2x^2-6x$ from $2x^2-5x$. Change the signs and then add. Then bring down the last term.	$\begin{array}{r} 2x \\ x-3\overline{)\ 2x^2-5x-3} \\ \underline{-2x^2+6x} \\ x-3 \end{array}$
Divide x by x. Put the answer, 1, in the quotient over the constant term.	$\begin{array}{r} 2x+1 \\ x-3\overline{)\ 2x^2-5x-3} \\ \underline{-2x^2-(-6x)} \\ x-3 \end{array}$
Multiply 1 times $x-3$.	$\begin{array}{r} 2x+1 \\ x-3\overline{)\ 2x^2-5x-3} \\ \underline{-2x^2+6x} \\ x-3 \\ \underline{x-3} \end{array}$
Subtract $x-3$ from $x-3$ by changing the signs and adding.	$\begin{array}{r} 2x+1 \\ x-3\overline{)\ 2x^2-5x-3} \\ \underline{-2x^2+6x} \\ x-3 \\ \underline{-x+3} \\ 0 \end{array}$
To check, multiply $(x-3)(2x+1)$.	
The result should be $2x^2-5x-3$.	

TRY IT : : 6.169 Find the quotient: $\left(2x^2-3x-20\right)\div(x-4)$.

TRY IT : : 6.170 Find the quotient: $\left(3x^2-16x-12\right)\div(x-6)$.

When we divided 875 by 25, we had no remainder. But sometimes division of numbers does leave a remainder. The same is true when we divide polynomials. In **Example 6.86**, we'll have a division that leaves a remainder. We write the remainder as a fraction with the divisor as the denominator.

EXAMPLE 6.86

Find the quotient: $\left(x^3-x^2+x+4\right)\div(x+1)$.

Solution

	$(x^3 - x^2 + x + 4) \div (x + 1)$
Write it as a long division problem.	
Be sure the dividend is in standard form.	$x+1 \overline{)\, x^3 - x^2 + x + 4}$
Divide x^3 by x. Put the answer, x^2, in the quotient over the x^2 term. Multiply x^2 times $x + 1$. Line up the like terms under the dividend.	$\begin{array}{r} x^2 \\ x+1 \overline{)\, x^3 - x^2 + x + 4} \\ \underline{x^3 + x^2} \end{array}$
Subtract $x^3 + x^2$ from $x^3 - x^2$ by changing the signs and adding. Then bring down the next term.	$\begin{array}{r} x^2 \\ x+1 \overline{)\, x^3 - x^2 + x + 4} \\ \underline{-x^3 + (-x^2)} \\ -2x^2 + x \end{array}$
Divide $-2x^2$ by x. Put the answer, $-2x$, in the quotient over the x term. Multiply $-2x$ times $x + 1$. Line up the like terms under the dividend.	$\begin{array}{r} x^2 - 2x \\ x+1 \overline{)\, x^3 - x^2 + x + 4} \\ \underline{-x^3 + (-x^2)} \\ -2x^2 + x \\ \underline{-2x^2 - 2x} \end{array}$
Subtract $-2x^2 - 2x$ from $-2x^2 + x$ by changing the signs and adding. Then bring down the last term.	$\begin{array}{r} x^2 - 2x \\ x+1 \overline{)\, x^3 - x^2 + x + 4} \\ \underline{-x^3 + (-x^2)} \\ -2x^2 + x \\ \underline{+2x^2 + 2x} \\ 3x + 4 \end{array}$
Divide $3x$ by x. Put the answer, 3, in the quotient over the constant term. Multiply 3 times $x + 1$. Line up the like terms under the dividend.	$\begin{array}{r} x^2 - 2x + 3 \\ x+1 \overline{)\, x^3 - x^2 + x + 4} \\ \underline{-x^3 + (-x^2)} \\ -2x^2 + x \\ \underline{+2x^2 + 2x} \\ 3x + 4 \\ \underline{3x + 3} \end{array}$
Subtract $3x + 3$ from $3x + 4$ by changing the signs and adding. Write the remainder as a fraction with the divisor as the denominator.	$\begin{array}{r} x^2 - 2x + 3 + \frac{1}{x+1} \\ x+1 \overline{)\, x^3 - x^2 + x + 4} \\ \underline{-x^3 + (-x^2)} \\ -2x^2 + x \\ \underline{+2x^2 + 2x} \\ 3x + 4 \\ \underline{-3x + (-3)} \\ 1 \end{array}$
To check, multiply $(x+1)\left(x^2 - 2x + 3 + \frac{1}{x+1}\right)$. The result should be $x^3 - x^2 + x + 4$.	

TRY IT : : 6.171 Find the quotient: $\left(x^3 + 5x^2 + 8x + 6\right) \div (x + 2)$.

TRY IT : : 6.172 Find the quotient: $\left(2x^3 + 8x^2 + x - 8\right) \div (x + 1)$.

Look back at the dividends in **Example 6.84**, **Example 6.85**, and **Example 6.86**. The terms were written in descending order of degrees, and there were no missing degrees. The dividend in **Example 6.87** will be $x^4 - x^2 + 5x - 2$. It is missing an x^3 term. We will add in $0x^3$ as a placeholder.

EXAMPLE 6.87

Find the quotient: $\left(x^4 - x^2 + 5x - 2\right) \div (x + 2)$.

Solution

Notice that there is no x^3 term in the dividend. We will add $0x^3$ as a placeholder.

	$(x^4 - x^2 + 5x - 2) \div (x + 2)$
Write it as a long division problem. Be sure the dividend is in standard form with placeholders for missing terms.	$x+2\overline{)\,x^4 - 0x^3 - x^2 + 5x - 2}$
Divide x^4 by x. Put the answer, x^3, in the quotient over the x^3 term. Multiply x^3 times $x + 2$. Line up the like terms. Subtract and then bring down the next term.	$\begin{array}{r} x^3 \\ x+2\overline{)\,x^4+0x^3-x^2+5x-2} \\ \underline{-(x^4+2x^3)} \\ -2x^3-x^2 \end{array}$ It may be helpful to change the signs and add.
Divide $-2x^3$ by x. Put the answer, $-2x^2$, in the quotient over the x^2 term. Multiply $-2x^2$ times $x + 1$. Line up the like terms. Subtract and bring down the next term.	$\begin{array}{r} x^3-2x^2 \\ x+2\overline{)\,x^4+0x^3-x^2+5x-2} \\ \underline{-(x^4+2x^3)} \\ -2x^3-x^2 \\ \underline{-(-2x^3-4x^2)} \\ 3x^2+5x \end{array}$ It may be helpful to change the signs and add.
Divide $3x^2$ by x. Put the answer, $3x$, in the quotient over the x term. Multiply $3x$ times $x + 1$. Line up the like terms. Subtract and bring down the next term.	$\begin{array}{r} x^3-2x^2+3x \\ x+2\overline{)\,x^4+0x^3-x^2+5x-2} \\ \underline{-(x^4+2x^3)} \\ -2x^3-x^2 \\ \underline{-(-2x^3-4x^2)} \\ 3x^2+5x \\ \underline{-(3x^2+6x)} \\ -x-2 \end{array}$ It may be helpful to change the signs and add.
Divide $-x$ by x. Put the answer, -1, in the quotient over the constant term. Multiply -1 times $x + 1$. Line up the like terms. Change the signs, add.	$\begin{array}{r} x^3-2x^2+3x-1 \\ x+2\overline{)\,x^4+0x^3-x^2+5x-2} \\ \underline{-(x^4+2x^3)} \\ -2x^3-x^2 \\ \underline{-(-2x^3-4x^2)} \\ 3x^2+5x \\ \underline{-(3x^2+6x)} \\ -x-2 \\ \underline{-(-x-2)} \\ 0 \end{array}$ It may be helpful to change the signs and add.
To check, multiply $(x + 2)\left(x^3 - 2x^2 + 3x - 1\right)$.	

The result should be $x^4 - x^2 + 5x - 2$.

> **TRY IT : :** 6.173 Find the quotient: $\left(x^3 + 3x + 14\right) \div (x + 2)$.

> **TRY IT : :** 6.174 Find the quotient: $\left(x^4 - 3x^3 - 1000\right) \div (x + 5)$.

In **Example 6.88**, we will divide by $2a - 3$. As we divide we will have to consider the constants as well as the variables.

EXAMPLE 6.88

Find the quotient: $\left(8a^3 + 27\right) \div (2a + 3)$.

Solution

This time we will show the division all in one step. We need to add two placeholders in order to divide.

$$(8a^3 + 27) \div (2a + 3)$$

$$\begin{array}{rlll}
 & 4a^2 - 6a + 9 & & \\ \cline{2-2}
2a + 3 \big) & 8a^3 + 0a^2 + 0a + 27 & & \\
 & -(8a^3 + 12a^2) & \leftarrow & 4a^2(2a + 3) \\ \cline{2-2}
 & -12a^2 + 0a & & \\
 & -(-12a^2 - 18a) & \leftarrow & 6a(2a + 3) \\ \cline{2-2}
 & 18a + 27 & & \\
 & -(18a + 27) & \leftarrow & 9(2a + 3) \\ \cline{2-2}
 & 0 & &
\end{array}$$

To check, multiply $(2a + 3)\left(4a^2 - 6a + 9\right)$.

The result should be $8a^3 + 27$.

> **TRY IT : :** 6.175 Find the quotient: $\left(x^3 - 64\right) \div (x - 4)$.

> **TRY IT : :** 6.176 Find the quotient: $\left(125x^3 - 8\right) \div (5x - 2)$.

MEDIA : :

Access these online resources for additional instruction and practice with dividing polynomials:

- **Divide a Polynomial by a Monomial (https://openstax.org/l/25DividePolyMo1)**
- **Divide a Polynomial by a Monomial 2 (https://openstax.org/l/25DividePolyMo2)**
- **Divide Polynomial by Binomial (https://openstax.org/l/25DividePolyBin)**

6.6 EXERCISES

Practice Makes Perfect

In the following exercises, divide each polynomial by the monomial.

442. $\frac{45y + 36}{9}$

443. $\frac{30b + 75}{5}$

444. $\frac{8d^2 - 4d}{2}$

445. $\frac{42x^2 - 14x}{7}$

446. $\left(16y^2 - 20y\right) \div 4y$

447. $\left(55w^2 - 10w\right) \div 5w$

448. $\left(9n^4 + 6n^3\right) \div 3n$

449. $\left(8x^3 + 6x^2\right) \div 2x$

450. $\frac{18y^2 - 12y}{-6}$

451. $\frac{20b^2 - 12b}{-4}$

452. $\frac{35a^4 + 65a^2}{-5}$

453. $\frac{51m^4 + 72m^3}{-3}$

454. $\frac{310y^4 - 200y^3}{5y^2}$

455. $\frac{412z^8 - 48z^5}{4z^3}$

456. $\frac{46x^3 + 38x^2}{2x^2}$

457. $\frac{51y^4 + 42y^2}{3y^2}$

458. $\left(24p^2 - 33p\right) \div (-3p)$

459. $\left(35x^4 - 21x\right) \div (-7x)$

460. $\left(63m^4 - 42m^3\right) \div \left(-7m^2\right)$

461. $\left(48y^4 - 24y^3\right) \div \left(-8y^2\right)$

462. $\left(63a^2b^3 + 72ab^4\right) \div (9ab)$

463. $\left(45x^3y^4 + 60xy^2\right) \div (5xy)$

464. $\frac{52p^5q^4 + 36p^4q^3 - 64p^3q^2}{4p^2q}$

465. $\frac{49c^2d^2 - 70c^3d^3 - 35c^2d^4}{7cd^2}$

466. $\frac{66x^3y^2 - 110x^2y^3 - 44x^4y^3}{11x^2y^2}$

467. $\frac{72r^5s^2 + 132r^4s^3 - 96r^3s^5}{12r^2s^2}$

468. $\frac{4w^2 + 2w - 5}{2w}$

469. $\frac{12q^2 + 3q - 1}{3q}$

470. $\frac{10x^2 + 5x - 4}{-5x}$

471. $\frac{20y^2 + 12y - 1}{-4y}$

472. $\frac{36p^3 + 18p^2 - 12p}{6p^2}$

473. $\frac{63a^3 - 108a^2 + 99a}{9a^2}$

Divide a Polynomial by a Binomial

In the following exercises, divide each polynomial by the binomial.

474. $\left(y^2 + 7y + 12\right) \div (y + 3)$

475. $\left(d^2 + 8d + 12\right) \div (d + 2)$

476. $\left(x^2 - 3x - 10\right) \div (x + 2)$

477. $\left(a^2 - 2a - 35\right) \div (a + 5)$

478. $\left(t^2 - 12t + 36\right) \div (t - 6)$

479. $\left(x^2 - 14x + 49\right) \div (x - 7)$

480. $\left(6m^2 - 19m - 20\right) \div (m - 4)$

481. $\left(4x^2 - 17x - 15\right) \div (x - 5)$

482. $\left(q^2 + 2q + 20\right) \div (q + 6)$

483. $\left(p^2 + 11p + 16\right) \div (p + 8)$

484. $\left(y^2 - 3y - 15\right) \div (y - 8)$

485. $\left(x^2 + 2x - 30\right) \div (x - 5)$

486. $\left(3b^3 + b^2 + 2\right) \div (b + 1)$

487. $\left(2n^3 - 10n + 24\right) \div (n + 3)$

488. $\left(2y^3 - 6y - 36\right) \div (y - 3)$

489. $\left(7q^3 - 5q - 2\right) \div (q - 1)$

490. $\left(z^3 + 1\right) \div (z + 1)$

491. $\left(m^3 + 1000\right) \div (m + 10)$

492. $\left(a^3 - 125\right) \div (a - 5)$

493. $\left(x^3 - 216\right) \div (x - 6)$

494. $\left(64x^3 - 27\right) \div (4x - 3)$

495. $\left(125y^3 - 64\right) \div (5y - 4)$

Everyday Math

496. Average cost Pictures Plus produces digital albums. The company's average cost (in dollars) to make x albums is given by the expression $\frac{7x + 500}{x}$.

ⓐ Find the quotient by dividing the numerator by the denominator.

ⓑ What will the average cost (in dollars) be to produce 20 albums?

497. Handshakes At a company meeting, every employee shakes hands with every other employee. The number of handshakes is given by the expression $\frac{n^2 - n}{2}$, where n represents the number of employees. How many handshakes will there be if there are 10 employees at the meeting?

Writing Exercises

498. James divides $48y + 6$ by 6 this way: $\frac{48y + \cancel{6}}{\cancel{6}} = 48y$. What is wrong with his reasoning?

499. Divide $\frac{10x^2 + x - 12}{2x}$ and explain with words how you get each term of the quotient.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
divide a polynomial by a monomial.			
divide a polynomial by a binomial.			

ⓑ *After reviewing this checklist, what will you do to become confident for all goals?*

6.7 Integer Exponents and Scientific Notation

Learning Objectives

By the end of this section, you will be able to:

- Use the definition of a negative exponent
- Simplify expressions with integer exponents
- Convert from decimal notation to scientific notation
- Convert scientific notation to decimal form
- Multiply and divide using scientific notation

Be Prepared!

Before you get started, take this readiness quiz.

1. What is the place value of the 6 in the number $64{,}891$?
   If you missed this problem, review **Example 1.1**.
2. Name the decimal: 0.0012.
   If you missed this problem, review **Example 1.91**.
3. Subtract: $5-(-3)$.
   If you missed this problem, review **Example 1.42**.

Use the Definition of a Negative Exponent

We saw that the Quotient Property for Exponents introduced earlier in this chapter, has two forms depending on whether the exponent is larger in the numerator or the denominator.

Quotient Property for Exponents

If a is a real number, $a \neq 0$, and m and n are whole numbers, then

$$\frac{a^m}{a^n}=a^{m-n},\ m>n \quad \text{and} \quad \frac{a^m}{a^n}=\frac{1}{a^{n-m}},\ n>m$$

What if we just subtract exponents regardless of which is larger?

Let's consider $\frac{x^2}{x^5}$.

We subtract the exponent in the denominator from the exponent in the numerator.

$$\frac{x^2}{x^5}$$

$$x^{2-5}$$

$$x^{-3}$$

We can also simplify $\frac{x^2}{x^5}$ by dividing out common factors:

$$\frac{\cancel{x}\cdot\cancel{x}}{\cancel{x}\cdot\cancel{x}\cdot x\cdot x\cdot x}$$

$$\frac{1}{x^3}$$

This implies that $x^{-3}=\frac{1}{x^3}$ and it leads us to the definition of a *negative exponent*.

Negative Exponent

If n is an integer and $a \neq 0$, then $a^{-n}=\frac{1}{a^n}$.

The negative exponent tells us we can re-write the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

EXAMPLE 6.89

Simplify: ⓐ 4^{-2} ⓑ 10^{-3}.

Solution

ⓐ

	4^{-2}
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{4^2}$
Simplify.	$\frac{1}{16}$

ⓑ

	10^{-3}
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{10^3}$
Simplify.	$\frac{1}{1000}$

TRY IT : : 6.177 Simplify: ⓐ 2^{-3} ⓑ 10^{-7}.

TRY IT : : 6.178 Simplify: ⓐ 3^{-2} ⓑ 10^{-4}.

In **Example 6.89** we raised an integer to a negative exponent. What happens when we raise a fraction to a negative exponent? We'll start by looking at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent.

	$\frac{1}{a^{-n}}$
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\frac{1}{a^n}}$
Simplify the complex fraction.	$1 \cdot \frac{a^n}{1}$
Multiply.	a^n

This leads to the Property of Negative Exponents.

Property of Negative Exponents

If n is an integer and $a \neq 0$, then $\frac{1}{a^{-n}} = a^n$.

EXAMPLE 6.90

Simplify: ⓐ $\frac{1}{y^{-4}}$ ⓑ $\frac{1}{3^{-2}}$.

Solution

ⓐ

	$\dfrac{1}{y^{-4}}$
Use the property of a negative exponent, $\dfrac{1}{a^{-n}} = a^n$.	y^4

ⓑ

	$\dfrac{1}{3^{-2}}$
Use the property of a negative exponent, $\dfrac{1}{a^{-n}} = a^n$.	3^2
Simplify.	9

TRY IT : : 6.179 Simplify: ⓐ $\dfrac{1}{p^{-8}}$ ⓑ $\dfrac{1}{4^{-3}}$.

TRY IT : : 6.180 Simplify: ⓐ $\dfrac{1}{q^{-7}}$ ⓑ $\dfrac{1}{2^{-4}}$.

Suppose now we have a fraction raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

	$\left(\dfrac{3}{4}\right)^{-2}$
Use the definition of a ne ative exponent, $a^{-n} = \dfrac{1}{a^n}$.	$\dfrac{1}{\left(\frac{3}{4}\right)^2}$
Simplify the denominator.	$\dfrac{1}{\frac{9}{16}}$
Simplify the complex fraction.	$\dfrac{16}{9}$
But we know that $\dfrac{16}{9}$ is $\left(\dfrac{4}{3}\right)^2$.	
This tells us that:	$\left(\dfrac{3}{4}\right)^{-2} = \left(\dfrac{4}{3}\right)^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the *Quotient to a Negative Power Property*.

Quotient to a Negative Exponent Property

If a and b are real numbers, $a \neq 0,\ b \neq 0,$ and n is an integer, then $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$.

EXAMPLE 6.91

Simplify: ⓐ $\left(\dfrac{5}{7}\right)^{-2}$ ⓑ $\left(-\dfrac{2x}{y}\right)^{-3}$.

Solution

ⓐ

	$\left(\frac{5}{7}\right)^{-2}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$\left(\frac{7}{5}\right)^{2}$
Simplify.	$\frac{49}{25}$

ⓑ

	$\left(-\frac{2x}{y}\right)^{-3}$
Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$\left(-\frac{y}{2x}\right)^{3}$
Simplify.	$-\frac{y^3}{8x^3}$

TRY IT : : 6.181 Simplify: ⓐ $\left(\frac{2}{3}\right)^{-4}$ ⓑ $\left(-\frac{6m}{n}\right)^{-2}$.

TRY IT : : 6.182 Simplify: ⓐ $\left(\frac{3}{5}\right)^{-3}$ ⓑ $\left(-\frac{a}{2b}\right)^{-4}$.

When simplifying an expression with exponents, we must be careful to correctly identify the base.

EXAMPLE 6.92

Simplify: ⓐ $(-3)^{-2}$ ⓑ -3^{-2} ⓒ $\left(-\frac{1}{3}\right)^{-2}$ ⓓ $-\left(\frac{1}{3}\right)^{-2}$.

Solution

ⓐ Here the exponent applies to the base -3.

	$(-3)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$\frac{1}{(-3)^{-2}}$
Simplify.	$\frac{1}{9}$

ⓑ The expression -3^{-2} means "find the opposite of 3^{-2}". Here the exponent applies to the base 3.

	-3^{-2}
Rewrite as a product with -1.	$-1 \cdot 3^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$-1 \cdot \frac{1}{3^2}$
Simplify.	$-\frac{1}{9}$

ⓒ Here the exponent applies to the base $\left(-\frac{1}{3}\right)$.

	$\left(-\frac{1}{3}\right)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$\left(-\frac{3}{1}\right)^{2}$
Simplify.	9

ⓓ The expression $-\left(\frac{1}{3}\right)^{-2}$ means "find the opposite of $\left(\frac{1}{3}\right)^{-2}$". Here the exponent applies to the base $\left(\frac{1}{3}\right)$.

	$-\left(\frac{1}{3}\right)^{-2}$
Rewrite as a product with -1.	$-1\cdot\left(\frac{1}{3}\right)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$-1\cdot\left(\frac{3}{1}\right)^{2}$
Simplify.	-9

TRY IT : : 6.183 Simplify: ⓐ $(-5)^{-2}$ ⓑ -5^{-2} ⓒ $\left(-\frac{1}{5}\right)^{-2}$ ⓓ $-\left(\frac{1}{5}\right)^{-2}$.

TRY IT : : 6.184 Simplify: ⓐ $(-7)^{-2}$ ⓑ -7^{-2}, ⓒ $\left(-\frac{1}{7}\right)^{-2}$ ⓓ $-\left(\frac{1}{7}\right)^{-2}$.

We must be careful to follow the Order of Operations. In the next example, parts (a) and (b) look similar, but the results are different.

EXAMPLE 6.93

Simplify: ⓐ $4\cdot 2^{-1}$ ⓑ $(4\cdot 2)^{-1}$.

Solution

ⓐ

Do exponents before multiplication.	$4\cdot 2^{-1}$
Use $a^{-n}=\frac{1}{a^n}$.	$4\cdot\frac{1}{2^1}$
Simplify.	2

ⓑ

	$(4\cdot 2)^{-1}$
Simplify inside the parentheses fir t.	$(8)^{-1}$
Use $a^{-n}=\frac{1}{a^n}$.	$\frac{1}{8^1}$
Simplify.	$\frac{1}{8}$

TRY IT : : 6.185 Simplify: ⓐ $6\cdot 3^{-1}$ ⓑ $(6\cdot 3)^{-1}$.

TRY IT : : 6.186 Simplify: ⓐ $8\cdot 2^{-2}$ ⓑ $(8\cdot 2)^{-2}$.

When a variable is raised to a negative exponent, we apply the definition the same way we did with numbers. We will assume all variables are non-zero.

EXAMPLE 6.94

Simplify: ⓐ x^{-6} ⓑ $\left(u^4\right)^{-3}$.

Solution

ⓐ

	x^{-6}
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{x^6}$

ⓑ

	$\left(u^4\right)^{-3}$
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\left(u^4\right)^3}$
Simplify.	$\frac{1}{u^{12}}$

TRY IT : : 6.187 Simplify: ⓐ y^{-7} ⓑ $\left(z^3\right)^{-5}$.

TRY IT : : 6.188 Simplify: ⓐ p^{-9} ⓑ $\left(q^4\right)^{-6}$.

When there is a product and an exponent we have to be careful to apply the exponent to the correct quantity. According to the Order of Operations, we simplify expressions in parentheses before applying exponents. We'll see how this works in the next example.

EXAMPLE 6.95

Simplify: ⓐ $5y^{-1}$ ⓑ $(5y)^{-1}$ ⓒ $(-5y)^{-1}$.

Solution

ⓐ

	$5y^{-1}$
Notice the exponent applies to just the base y. Take the reciprocal of y and change the sign of the exponent.	$5 \cdot \frac{1}{y^1}$
Simplify.	$\frac{5}{y}$

ⓑ

	$(5y)^{-1}$
Here the parentheses make the exponent apply to the base $5y$. Take the reciprocal of $5y$ and change the sign of the exponent.	$\frac{1}{(5y)^1}$
Simplify.	$\frac{1}{5y}$

ⓒ

	$(-5y)^{-1}$
The base here is $-5y$. Take the reciprocal of $-5y$ and change the sign of the exponent.	$\frac{1}{(-5y)^1}$
Simplify.	$\frac{1}{-5y}$
Use $\frac{a}{-b} = -\frac{a}{b}$.	$-\frac{1}{5y}$

TRY IT : : 6.189 Simplify: ⓐ $8p^{-1}$ ⓑ $(8p)^{-1}$ ⓒ $(-8p)^{-1}$.

TRY IT : : 6.190 Simplify: ⓐ $11q^{-1}$ ⓑ $(11q)^{-1}$ $-(11q)^{-1}$ ⓒ $(-11q)^{-1}$.

With negative exponents, the Quotient Rule needs only one form $\frac{a^m}{a^n} = a^{m-n}$, for $a \neq 0$. When the exponent in the denominator is larger than the exponent in the numerator, the exponent of the quotient will be negative.

Simplify Expressions with Integer Exponents

All of the exponent properties we developed earlier in the chapter with whole number exponents apply to integer exponents, too. We restate them here for reference.

Summary of Exponent Properties

If a and b are real numbers, and m and n are integers, then

Product Property	$a^m \cdot a^n$	$=$	a^{m+n}
Power Property	$(a^m)^n$	$=$	$a^{m \cdot n}$
Product to a Power	$(ab)^m$	$=$	$a^m b^m$
Quotient Property	$\frac{a^m}{a^n}$	$=$	$a^{m-n}, a \neq 0$
Zero Exponent Property	a^0	$=$	$1, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m$	$=$	$\frac{a^m}{b^m}, \quad b \neq 0$
Properties of Negative Exponents	a^{-n}	$=$	$\frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n}$	$=$	$\left(\frac{b}{a}\right)^n$

EXAMPLE 6.96

Simplify: ⓐ $x^{-4} \cdot x^6$ ⓑ $y^{-6} \cdot y^4$ ⓒ $z^{-5} \cdot z^{-3}$.

Solution

ⓐ

	$x^{-4} \cdot x^6$
Use the Product Property, $a^m \cdot a^n = a^{m+n}$.	x^{-4+6}
Simplify.	x^2

ⓑ

	$y^{-6} \cdot y^4$
Notice the same bases, so add the exponents.	y^{-6+4}
Simplify.	y^{-2}
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{y^2}$

ⓒ

	$z^{-5} \cdot z^{-3}$
Add the exponents, since the bases are the same.	z^{-5-3}
Simplify.	z^{-8}
Take the reciprocal and change the sign of the exponent, using the definition of a ne ative exponent.	$\frac{1}{z^8}$

TRY IT : : 6.191 Simplify: ⓐ $x^{-3} \cdot x^7$ ⓑ $y^{-7} \cdot y^2$ ⓒ $z^{-4} \cdot z^{-5}$.

TRY IT : : 6.192 Simplify: ⓐ $a^{-1} \cdot a^6$ ⓑ $b^{-8} \cdot b^4$ ⓒ $c^{-8} \cdot c^{-7}$.

In the next two examples, we'll start by using the Commutative Property to group the same variables together. This makes it easier to identify the like bases before using the Product Property.

EXAMPLE 6.97

Simplify: $\left(m^4 n^{-3}\right)\left(m^{-5} n^{-2}\right)$.

Solution

	$\left(m^4 n^{-3}\right)\left(m^{-5} n^{-2}\right)$
Use the Commutative Property to get like bases together.	$m^4 m^{-5} \cdot n^{-2} n^{-3}$
Add the exponents for each base.	$m^{-1} \cdot n^{-5}$
Take reciprocals and change the signs of the exponents.	$\frac{1}{m^1} \cdot \frac{1}{n^5}$
Simplify.	$\frac{1}{mn^5}$

TRY IT : : 6.193 Simplify: $\left(p^6 q^{-2}\right)\left(p^{-9} q^{-1}\right)$.

TRY IT : : 6.194 Simplify: $\left(r^5 s^{-3}\right)\left(r^{-7} s^{-5}\right)$.

If the monomials have numerical coefficients, we multiply the coefficients, just like we did earlier.

EXAMPLE 6.98

Simplify: $\left(2x^{-6} y^8\right)\left(-5x^5 y^{-3}\right)$.

Solution

	$\left(2x^{-6}y^{8}\right)\left(-5x^{5}y^{-3}\right)$
Rewrite with the like bases together.	$2(-5)\cdot\left(x^{-6}x^{5}\right)\cdot\left(y^{8}y^{-3}\right)$
Multiply the coefficients and add he exponents of each variable.	$-10\cdot x^{-1}\cdot y^{5}$
Use the definition of a ne ative exponent, $a^{-n}=\frac{1}{a^{n}}$.	$-10\cdot\frac{1}{x^{1}}\cdot y^{5}$
Simplify.	$\frac{-10y^{5}}{x}$

TRY IT : : 6.195 Simplify: $\left(3u^{-5}v^{7}\right)\left(-4u^{4}v^{-2}\right)$.

TRY IT : : 6.196 Simplify: $\left(-6c^{-6}d^{4}\right)\left(-5c^{-2}d^{-1}\right)$.

In the next two examples, we'll use the Power Property and the Product to a Power Property.

EXAMPLE 6.99

Simplify: $\left(6k^{3}\right)^{-2}$.

Solution

	$\left(6k^{3}\right)^{-2}$
Use the Product to a Power Property, $(ab)^{m}=a^{m}b^{m}$.	$(6)^{-2}\left(k^{3}\right)^{-2}$
Use the Power Property, $(a^{m})^{n}=a^{m\cdot n}$.	$6^{-2}k^{-6}$
Use the Definition of a egative Exponent, $a^{-n}=\frac{1}{a^{n}}$.	$\frac{1}{6^{2}}\cdot\frac{1}{k^{6}}$
Simplify.	$\frac{1}{36k^{6}}$

TRY IT : : 6.197 Simplify: $\left(-4x^{4}\right)^{-2}$.

TRY IT : : 6.198 Simplify: $\left(2b^{3}\right)^{-4}$.

EXAMPLE 6.100

Simplify: $\left(5x^{-3}\right)^{2}$.

Solution

	$\left(5x^{-3}\right)^2$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$5^2\left(x^{-3}\right)^2$
Simplify 5^2 and multiply the exponents of x using the Power Property, $(a^m)^n = a^{m \cdot n}$.	$25 \cdot x^{-6}$
Rewrite x^{-6} by using the Definition of a egative Exponent, $a^{-n} = \frac{1}{a^n}$.	$25 \cdot \frac{1}{x^6}$
Simplify.	$\frac{25}{x^6}$

TRY IT : : 6.199 Simplify: $\left(8a^{-4}\right)^2$.

TRY IT : : 6.200 Simplify: $\left(2c^{-4}\right)^3$.

To simplify a fraction, we use the Quotient Property and subtract the exponents.

EXAMPLE 6.101

Simplify: $\frac{r^5}{r^{-4}}$.

Solution

	$\frac{r^5}{r^{-4}}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	$r^{5-(-4)}$
Simplify.	r^9

TRY IT : : 6.201 Simplify: $\frac{x^8}{x^{-3}}$.

TRY IT : : 6.202 Simplify: $\frac{y^8}{y^{-6}}$.

Convert from Decimal Notation to Scientific Notation

Remember working with place value for whole numbers and decimals? Our number system is based on powers of 10. We use tens, hundreds, thousands, and so on. Our decimal numbers are also based on powers of tens—tenths, hundredths, thousandths, and so on. Consider the numbers 4,000 and 0.004 . We know that 4,000 means $4 \times 1{,}000$ and 0.004 means $4 \times \frac{1}{1{,}000}$.

If we write the 1000 as a power of ten in exponential form, we can rewrite these numbers in this way:

$$\begin{array}{ll} 4{,}000 & 0.004 \\ 4 \times 1{,}000 & 4 \times \dfrac{1}{1{,}000} \\ 4 \times 10^3 & 4 \times \dfrac{1}{10^3} \\ & 4 \times 10^{-3} \end{array}$$

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than 10, and the second factor is a power of 10 written in exponential form, it is said to be in *scientific notation.*

Scientific Notation

A number is expressed in **scientific notation** when it is of the form

$$a \times 10^n \text{ where } 1 \le a < 10 \text{ and } n \text{ is an integer}$$

It is customary in scientific notation to use as the $\times$ multiplication sign, even though we avoid using this sign elsewhere in algebra.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to scientific notation.

$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
Moved the decimal point 3 places to the left.	Moved the decimal point 3 places to the right.

In both cases, the decimal was moved 3 places to get the first factor between 1 and 10.

The power of 10 is positive when the number is larger than 1:	$4{,}000 = 4 \times 10^3$
The power of 10 is negative when the number is between 0 and 1:	$0.004 = 4 \times 10^{-3}$

EXAMPLE 6.102 HOW TO CONVERT FROM DECIMAL NOTATION TO SCIENTIFIC NOTATION

Write in scientific notation: 37,000.

Solution

Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.	Remember, there is a decimal at the end of 37,000. Move the decimal after the 3. 3.700 is between 1 and 10.	37,000.
Step 2. Count the number of decimal places, *n*, that the decimal point was moved.	The decimal point was moved 4 places to the left.	37000.
Step 3. Write the number as a product with a power of 10. If the original number is: Greater than 1, the power of 10 will be 10^n. Between 0 and 1, the power of 10 will be 10^{-n}.	37,000 is greater than 1 so the power of 10 will have exponent 4.	3.7×10^4
Step 4. Check.	Check to see if your answer makes sense.	10^4 is 10,000 and 10,000 times 3.7 will be 37,000. $37{,}000 = 3.7 \times 10^4$

TRY IT : : 6.203 Write in scientific notation: 96,000.

TRY IT : : 6.204 Write in scientific notation: $48{,}300$.

HOW TO : : CONVERT FROM DECIMAL NOTATION TO SCIENTIFIC NOTATION

Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.

Step 2. Count the number of decimal places, n, that the decimal point was moved.

Step 3. Write the number as a product with a power of 10.
If the original number is:

- greater than 1, the power of 10 will be 10^n.
- between 0 and 1, the power of 10 will be 10^{-n}.

Step 4. Check.

EXAMPLE 6.103

Write in scientific notation: 0.0052.

Solution

The original number, 0.0052, is between 0 and 1 so we will have a negative power of 10.

	0.0052
Move the decimal point to get 5.2, a number between 1 and 10.	0.0052
Count the number of decimal places the point was moved.	3 places
Write as a product with a power of 10.	5.2×10^{-3}
Check.	

$$5.2 \times 10^{-3}$$
$$5.2 \times \frac{1}{10^3}$$
$$5.2 \times \frac{1}{1000}$$
$$5.2 \times 0.001$$

0.0052	$0.0052 = 5.2 \times 10^{-3}$

TRY IT : : 6.205 Write in scientific notation: 0.0078.

TRY IT : : 6.206 Write in scientific notation: 0.0129.

Convert Scientific Notation to Decimal Form

How can we convert from scientific notation to decimal form? Let's look at two numbers written in scientific notation and see.

$$\begin{array}{cc} 9.12 \times 10^4 & 9.12 \times 10^{-4} \\ 9.12 \times 10{,}000 & 9.12 \times 0.0001 \\ 91{,}200 & 0.000912 \end{array}$$

If we look at the location of the decimal point, we can see an easy method to convert a number from scientific notation to decimal form.

$9.12 \times 10^4 = 91{,}200$ $\qquad$ $9.12 \times 10^{-4} = 0.000912$

$9.12 \times 10^4 = 91{,}200$ $\qquad$ $9.12 \times 10^{-4} = 0.000912$

$9.12\_\_ \times 10^4 = 91{,}200$ $\qquad$ $\_\_\_9.12 \times 10^{-4} = 0.000912$

Move the decimal point 4 places to the right. $\qquad$ Move the decimal point 4 places to the left.

In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

EXAMPLE 6.104 HOW TO CONVERT SCIENTIFIC NOTATION TO DECIMAL FORM

Convert to decimal form: 6.2×10^3.

Solution

Step 1. Determine the exponent, *n*, on the factor 10.	The exponent is 3.	6.2×10^3
Step 2. Move the decimal *n* places, adding zeros if needed. If the exponent is positive, move the decimal point *n* places to the right. If the exponent is negative, move the decimal point $\lvert n \rvert$ places to the left.	The exponent is positive, so move the decimal point 3 places to the right. We need to add 2 zeros as placeholders.	6.200. 6,200
Step 3. Check to see if your answer makes sense.		10^3 is 1000 and 1000 times 6.2 will be 6,200. $6.2 \times 10^3 = 6{,}200$

TRY IT : : 6.207 Convert to decimal form: 1.3×10^3.

TRY IT : : 6.208 Convert to decimal form: 9.25×10^4.

The steps are summarized below.

HOW TO : : CONVERT SCIENTIFIC NOTATION TO DECIMAL FORM.

To convert scientific notation to decimal form:

Step 1. Determine the exponent, n, on the factor 10.

Step 2. Move the decimal n places, adding zeros if needed.

- If the exponent is positive, move the decimal point n places to the right.
- If the exponent is negative, move the decimal point $|n|$ places to the left.

Step 3. Check.

EXAMPLE 6.105

Convert to decimal form: 8.9×10^{-2}.

Solution

	8.9×10^{-2}
Determine the exponent, *n*, on the factor 10.	The exponent is −2.
Since the exponent is negative, move the decimal point 2 places to the left.	$.8.9$
Add zeros as needed for placeholders.	$8.9 \times 10^{-2} = 0.089$

TRY IT : : 6.209 Convert to decimal form: 1.2×10^{-4}.

TRY IT : : 6.210 Convert to decimal form: 7.5×10^{-2}.

Multiply and Divide Using Scientific Notation

Astronomers use very large numbers to describe distances in the universe and ages of stars and planets. Chemists use very small numbers to describe the size of an atom or the charge on an electron. When scientists perform calculations with very large or very small numbers, they use scientific notation. Scientific notation provides a way for the calculations to be done without writing a lot of zeros. We will see how the Properties of Exponents are used to multiply and divide numbers in scientific notation.

EXAMPLE 6.106

Multiply. Write answers in decimal form: $(4 \times 10^5)(2 \times 10^{-7})$.

Solution

	$(4 \times 10^5)(2 \times 10^{-7})$
Use the Commutative Property to rearrange the factors.	$4 \cdot 2 \cdot 10^5 \cdot 10^{-7}$
Multiply.	8×10^{-2}
Change to decimal form by moving the decimal two places left.	0.08

TRY IT : : 6.211 Multiply $(3 \times 10^6)(2 \times 10^{-8})$. Write answers in decimal form.

TRY IT : : 6.212 Multiply $(3 \times 10^{-2})(3 \times 10^{-1})$. Write answers in decimal form.

EXAMPLE 6.107

Divide. Write answers in decimal form: $\dfrac{9 \times 10^3}{3 \times 10^{-2}}$.

Solution

	$\dfrac{9 \times 10^3}{3 \times 10^{-2}}$
Separate the factors, rewriting as the product of two fractions.	$\dfrac{9}{3} \times \dfrac{10^3}{10^{-2}}$
Divide.	3×10^5
Change to decimal form by moving the decimal fi e places right.	300,000

TRY IT : : 6.213 Divide $\dfrac{8 \times 10^4}{2 \times 10^{-1}}$. Write answers in decimal form.

TRY IT : : 6.214 Divide $\dfrac{8 \times 10^2}{4 \times 10^{-2}}$. Write answers in decimal form.

MEDIA : :

Access these online resources for additional instruction and practice with integer exponents and scientific notation:

- **Negative Exponents (https://openstax.org/l/25Negexponents)**
- **Scientific Notation (https://openstax.org/l/25Scientnot1)**
- **Scientific Notation 2 (https://openstax.org/l/25Scientnot2)**

6.7 EXERCISES

Practice Makes Perfect

Use the Definition of a Negative Exponent

In the following exercises, simplify.

500.
ⓐ 4^{-2}
ⓑ 10^{-3}

501.
ⓐ 3^{-4}
ⓑ 10^{-2}

502.
ⓐ 5^{-3}
ⓑ 10^{-5}

503.
ⓐ 2^{-8}
ⓑ 10^{-2}

504.
ⓐ $\frac{1}{c^{-5}}$
ⓑ $\frac{1}{3^{-2}}$

505.
ⓐ $\frac{1}{c^{-5}}$
ⓑ $\frac{1}{5^{-2}}$

506.
ⓐ $\frac{1}{q^{-10}}$
ⓑ $\frac{1}{10^{-3}}$

507.
ⓐ $\frac{1}{t^{-9}}$
ⓑ $\frac{1}{10^{-4}}$

508.
ⓐ $\left(\frac{5}{8}\right)^{-2}$
ⓑ $\left(-\frac{3m}{n}\right)^{-2}$

509.
ⓐ $\left(\frac{3}{10}\right)^{-2}$
ⓑ $\left(-\frac{2}{cd}\right)^{-3}$

510.
ⓐ $\left(\frac{4}{9}\right)^{-3}$
ⓑ $\left(-\frac{u^2}{2v}\right)^{-5}$

511.
ⓐ $\left(\frac{7}{2}\right)^{-3}$
ⓑ $\left(-\frac{3}{xy^2}\right)^{-3}$

512.
ⓐ $(-5)^{-2}$
ⓑ -5^{-2}
ⓒ $\left(-\frac{1}{5}\right)^{-2}$
ⓓ $-\left(\frac{1}{5}\right)^{-2}$

513.
ⓐ $(-7)^{-2}$
ⓑ -7^{-2}
ⓒ $\left(-\frac{1}{7}\right)^{-2}$
ⓓ $-\left(\frac{1}{7}\right)^{-2}$

514.
ⓐ -3^{-3}
ⓑ $\left(-\frac{1}{3}\right)^{-3}$
ⓒ $-\left(\frac{1}{3}\right)^{-3}$
ⓓ $(-3)^{-3}$

515.
ⓐ -5^{-3}
ⓑ $\left(-\frac{1}{5}\right)^{-3}$
ⓒ $-\left(\frac{1}{5}\right)^{-3}$
ⓓ $(-5)^{-3}$

516.
ⓐ $3 \cdot 5^{-1}$
ⓑ $(3 \cdot 5)^{-1}$

517.
ⓐ $2 \cdot 5^{-1}$
ⓑ $(2 \cdot 5)^{-1}$

518.
ⓐ $4 \cdot 5^{-2}$
ⓑ $(4 \cdot 5)^{-2}$

519.
ⓐ $3 \cdot 4^{-2}$
ⓑ $(3 \cdot 4)^{-2}$

520.
ⓐ m^{-4}
ⓑ $\left(x^3\right)^{-4}$

521.
ⓐ b^{-5}
ⓑ $\left(k^2\right)^{-5}$

522.
ⓐ p^{-10}
ⓑ $\left(q^6\right)^{-8}$

523.
ⓐ s^{-8}
ⓑ $\left(a^9\right)^{-10}$

524.
ⓐ $7n^{-1}$
ⓑ $(7n)^{-1}$
ⓒ $(-7n)^{-1}$

525.
ⓐ $6r^{-1}$
ⓑ $(6r)^{-1}$
ⓒ $(-6r)^{-1}$

526.
ⓐ $(3p)^{-2}$
ⓑ $3p^{-2}$
ⓒ $-3p^{-2}$

527.
ⓐ $(2q)^{-4}$
ⓑ $2q^{-4}$
ⓒ $-2q^{-4}$

Simplify Expressions with Integer Exponents

In the following exercises, simplify.

528.
ⓐ $b^4 b^{-8}$
ⓑ $r^{-2} r^5$
ⓒ $x^{-7} x^{-3}$

529.
ⓐ $s^3 \cdot s^{-7}$
ⓑ $q^{-8} \cdot q^3$
ⓒ $y^{-2} \cdot y^{-5}$

530.
ⓐ $a^3 \cdot a^{-3}$
ⓑ $a \cdot a^3$
ⓒ $a \cdot a^{-3}$

531.
ⓐ $y^5 \cdot y^{-5}$
ⓑ $y \cdot y^5$
ⓒ $y \cdot y^{-5}$

532. $p^5 \cdot p^{-2} \cdot p^{-4}$

533. $x^4 \cdot x^{-2} \cdot x^{-3}$

534. $\left(w^4 x^{-5}\right)\left(w^{-2} x^{-4}\right)$

535. $\left(m^3 n^{-3}\right)\left(m^{-5} n^{-1}\right)$

536. $\left(uv^{-2}\right)\left(u^{-5} v^{-3}\right)$

537. $\left(pq^{-4}\right)\left(p^{-6} q^{-3}\right)$

538. $\left(-6c^{-3} d^9\right)\left(2c^4 d^{-5}\right)$

539. $\left(-2j^{-5} k^8\right)\left(7j^2 k^{-3}\right)$

540. $\left(-4r^{-2} s^{-8}\right)\left(9r^4 s^3\right)$

541. $\left(-5m^4 n^6\right)\left(8m^{-5} n^{-3}\right)$

542. $\left(5x^2\right)^{-2}$

543. $\left(4y^3\right)^{-3}$

544. $\left(3z^{-3}\right)^2$

545. $\left(2p^{-5}\right)^2$

546. $\dfrac{t^9}{t^{-3}}$

547. $\dfrac{n^5}{n^{-2}}$

548. $\dfrac{x^{-7}}{x^{-3}}$

549. $\dfrac{y^{-5}}{y^{-10}}$

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

550. 57,000

551. 340,000

552. 8,750,000

553. 1,290,000

554. 0.026

555. 0.041

556. 0.00000871

557. 0.00000103

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

558. 5.2×10^{2}

559. 8.3×10^{2}

560. 7.5×10^{6}

561. 1.6×10^{10}

562. 2.5×10^{-2}

563. 3.8×10^{-2}

564. 4.13×10^{-5}

565. 1.93×10^{-5}

Multiply and Divide Using Scientific Notation

In the following exercises, multiply. Write your answer in decimal form.

566. $\left(3 \times 10^{-5}\right)\left(3 \times 10^{9}\right)$

567. $\left(2 \times 10^{2}\right)\left(1 \times 10^{-4}\right)$

568. $\left(7.1 \times 10^{-2}\right)\left(2.4 \times 10^{-4}\right)$

569. $\left(3.5 \times 10^{-4}\right)\left(1.6 \times 10^{-2}\right)$

In the following exercises, divide. Write your answer in decimal form.

570. $\dfrac{7 \times 10^{-3}}{1 \times 10^{-7}}$

571. $\dfrac{5 \times 10^{-2}}{1 \times 10^{-10}}$

572. $\dfrac{6 \times 10^{4}}{3 \times 10^{-2}}$

573. $\dfrac{8 \times 10^{6}}{4 \times 10^{-1}}$

Everyday Math

574. The population of the United States on July 4, 2010 was almost 310,000,000. Write the number in scientific notation.

575. The population of the world on July 4, 2010 was more than 6,850,000,000. Write the number in scientific notation

576. The average width of a human hair is 0.0018 centimeters. Write the number in scientific notation.

577. The probability of winning the 2010 Megamillions lottery was about 0.0000000057. Write the number in scientific notation.

578. In 2010, the number of Facebook users each day who changed their status to 'engaged' was 2×10^{4}. Convert this number to decimal form.

579. At the start of 2012, the US federal budget had a deficit of more than $\$1.5 \times 10^{13}$. Convert this number to decimal form.

580. The concentration of carbon dioxide in the atmosphere is 3.9×10^{-4}. Convert this number to decimal form.

581. The width of a proton is 1×10^{-5} of the width of an atom. Convert this number to decimal form.

582. Health care costs The Centers for Medicare and Medicaid projects that consumers will spend more than $4 trillion on health care by 2017.

ⓐ Write 4 trillion in decimal notation.

ⓑ Write 4 trillion in scientific notation.

583. Coin production In 1942, the U.S. Mint produced 154,500,000 nickels. Write 154,500,000 in scientific notation.

584. Distance The distance between Earth and one of the brightest stars in the night star is 33.7 light years. One light year is about 6,000,000,000,000 (6 trillion), miles.

ⓐ Write the number of miles in one light year in scientific notation.

ⓑUse scientific notation to find the distance between Earth and the star in miles. Write the answer in scientific notation.

585. Debt At the end of fiscal year 2015 the gross United States federal government debt was estimated to be approximately $18,600,000,000,000 ($18.6 trillion), according to the Federal Budget. The population of the United States was approximately 300,000,000 people at the end of fiscal year 2015.

ⓐ Write the debt in scientific notation.

ⓑ Write the population in scientific notation.

ⓒ Find the amount of debt per person by using scientific notation to divide the debt by the population. Write the answer in scientific notation.

Writing Exercises

586.

ⓐ Explain the meaning of the exponent in the expression 2^3.

ⓑ Explain the meaning of the exponent in the expression 2^{-3}.

587. When you convert a number from decimal notation to scientific notation, how do you know if the exponent will be positive or negative?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the definition of a negative exponent.			
simplify expressions with integer exponents.			
convert from decimal notation to scientific notation.			
convert scientific notation to decimal form.			
multiply and divide using scientific notation.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

CHAPTER 6 REVIEW

KEY TERMS

binomial A binomial is a polynomial with exactly two terms.

conjugate pair A conjugate pair is two binomials of the form $(a-b), (a+b)$; the pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

degree of a constant The degree of any constant is 0.

degree of a polynomial The degree of a polynomial is the highest degree of all its terms.

degree of a term The degree of a term is the exponent of its variable.

monomial A monomial is a term of the form ax^m, where a is a constant and m is a whole number; a monomial has exactly one term.

negative exponent If n is a positive integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

polynomial A polynomial is a monomial, or two or more monomials combined by addition or subtraction.

scientific notation A number is expressed in scientific notation when it is of the form $a \times 10^n$ where $a \geq 1$ and $a < 10$ and n is an integer.

standard form A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees.

trinomial A trinomial is a polynomial with exactly three terms.

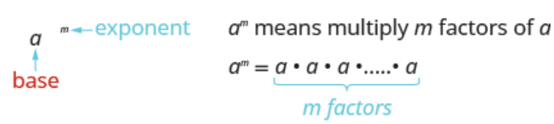
KEY CONCEPTS

6.1 Add and Subtract Polynomials

- **Monomials**
  - A monomial is a term of the form ax^m, where a is a constant and m is a whole number
- **Polynomials**
  - **polynomial**—A monomial, or two or more monomials combined by addition or subtraction is a polynomial.
  - **monomial**—A polynomial with exactly one term is called a monomial.
  - **binomial**—A polynomial with exactly two terms is called a binomial.
  - **trinomial**—A polynomial with exactly three terms is called a trinomial.
- **Degree of a Polynomial**
  - The **degree of a term** is the sum of the exponents of its variables.
  - The **degree of a constant** is 0.
  - The **degree of a polynomial** is the highest degree of all its terms.

6.2 Use Multiplication Properties of Exponents

- **Exponential Notation**



- **Properties of Exponents**
  - If a, b are real numbers and m, n are whole numbers, then

Product Property	$a^m \cdot a^n$	$=$	a^{m+n}
Power Property	$(a^m)^n$	$=$	$a^{m \cdot n}$
Product to a Power	$(ab)^m$	$=$	$a^m b^m$

6.3 Multiply Polynomials

- **FOIL Method for Multiplying Two Binomials**—To multiply two binomials:
  Step 1. Multiply the **First** terms.
  Step 2. Multiply the **Outer** terms.
  Step 3. Multiply the **Inner** terms.
  Step 4. Multiply the **Last** terms.

- **Multiplying Two Binomials**—To multiply binomials, use the:
  - Distributive Property (**Example 6.34**)
  - FOIL Method (**Example 6.39**)
  - Vertical Method (**Example 6.44**)

- **Multiplying a Trinomial by a Binomial**—To multiply a trinomial by a binomial, use the:
  - Distributive Property (**Example 6.45**)
  - Vertical Method (**Example 6.46**)

6.4 Special Products

- **Binomial Squares Pattern**
  - If a, b are real numbers,

$$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

  - $(a+b)^2 = a^2 + 2ab + b^2$
  - $(a-b)^2 = a^2 - 2ab + b^2$
  - To square a binomial: square the first term, square the last term, double their product.

- **Product of Conjugates Pattern**
  - If a, b are real numbers,

$$\underbrace{(a-b)(a+b)}_{\text{conjugates}} = \underset{\text{squares}}{a^2 \overset{\text{difference}}{-} b^2}$$

  - $(a-b)(a+b) = a^2 - b^2$
  - The product is called a difference of squares.

- **To multiply conjugates:**
  - **square the first term square the last term** write it as a difference of squares

6.5 Divide Monomials

- **Quotient Property for Exponents:**
  - If a is a real number, $a \neq 0$, and m, n are whole numbers, then:
    $\frac{a^m}{a^n} = a^{m-n},\ m > n$ and $\frac{a^m}{a^n} = \frac{1}{a^{m-n}},\ n > m$

- **Zero Exponent**
  - If a is a non-zero number, then $a^0 = 1$.

- **Quotient to a Power Property for Exponents**:
  - If a and b are real numbers, $b \neq 0,$ and m is a counting number, then:
    $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
  - To raise a fraction to a power, raise the numerator and denominator to that power.

- **Summary of Exponent Properties**
  - If a, b are real numbers and m, n are whole numbers, then

$$\begin{array}{lrcl}
\textbf{Product Property} & a^m \cdot a^n & = & a^{m+n} \\
\textbf{Power Property} & (a^m)^n & = & a^{m \cdot n} \\
\textbf{Product to a Power} & (ab)^m & = & a^m b^m \\
\textbf{Quotient Property} & \frac{a^m}{b^m} & = & a^{m-n}, \ a \neq 0, \ m > n \\
 & \frac{a^m}{a^n} & = & \frac{1}{a^{n-m}}, \ a \neq 0, \ n > m \\
\textbf{Zero Exponent Definitio} & a^o & = & 1, \ a \neq 0 \\
\textbf{Quotient to a Power Property} & \left(\frac{a}{b}\right)^m & = & \frac{a^m}{b^m}, \ b \neq 0
\end{array}$$

6.6 Divide Polynomials

- **Fraction Addition**
  - If $a, b,$ and c are numbers where $c \neq 0$, then
    $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ and $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

- **Division of a Polynomial by a Monomial**
  - To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

6.7 Integer Exponents and Scientific Notation

- **Property of Negative Exponents**
  - If n is a positive integer and $a \neq 0$, then $\frac{1}{a^{-n}} = a^n$
- **Quotient to a Negative Exponent**
  - If a, b are real numbers, $b \neq 0$ and n is an integer , then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$
- **To convert a decimal to scientific notation:**

  Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.

  Step 2. Count the number of decimal places, n, that the decimal point was moved.

  Step 3. Write the number as a product with a power of 10. If the original number is:
  - greater than 1, the power of 10 will be 10^n
  - between 0 and 1, the power of 10 will be 10^{-n}

  Step 4. Check.

- **To convert scientific notation to decimal form:**

  Step 1. Determine the exponent, n, on the factor 10.

Step 2. Move the decimal n places, adding zeros if needed.

- If the exponent is positive, move the decimal point n places to the right.
- If the exponent is negative, move the decimal point $|n|$ places to the left.

Step 3. Check.

REVIEW EXERCISES

6.1 Section 6.1 Add and Subtract Polynomials

Identify Polynomials, Monomials, Binomials and Trinomials

In the following exercises, determine if each of the following polynomials is a monomial, binomial, trinomial, or other polynomial.

588.
ⓐ $11c^4 - 23c^2 + 1$
ⓑ $9p^3 + 6p^2 - p - 5$
ⓒ $\frac{3}{7}x + \frac{5}{14}$
ⓓ 10
ⓔ $2y - 12$

589.
ⓐ $a^2 - b^2$
ⓑ $24d^3$
ⓒ $x^2 + 8x - 10$
ⓓ $m^2n^2 - 2mn + 6$
ⓔ $7y^3 + y^2 - 2y - 4$

Determine the Degree of Polynomials

In the following exercises, determine the degree of each polynomial.

590.
ⓐ $3x^2 + 9x + 10$
ⓑ $14a^2bc$
ⓒ $6y + 1$
ⓓ $n^3 - 4n^2 + 2n - 8$
ⓔ -19

591.
ⓐ $5p^3 - 8p^2 + 10p - 4$
ⓑ $-20q^4$
ⓒ $x^2 + 6x + 12$
ⓓ $23r^2s^2 - 4rs + 5$
ⓔ 100

Add and Subtract Monomials

In the following exercises, add or subtract the monomials.

592. $5y^3 + 8y^3$

593. $-14k + 19k$

594. $12q - (-6q)$

595. $-9c - 18c$

596. $12x - 4y - 9x$

597. $3m^2 + 7n^2 - 3m^2$

598. $6x^2y - 4x + 8xy^2$

599. $13a + b$

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

600. $(5x^2 + 12x + 1) + (6x^2 - 8x + 3)$

601. $(9p^2 - 5p + 3) + (4p^2 - 4)$

602. $(10m^2 - 8m - 1) - (5m^2 + m - 2)$

603. $(7y^2 - 8y) - (y - 4)$

604. Subtract $\left(3s^2 + 10\right)$ from $\left(15s^2 - 2s + 8\right)$

605. Find the sum of $\left(a^2 + 6a + 9\right)$ and $\left(5a^3 - 7\right)$

Evaluate a Polynomial for a Given Value of the Variable

In the following exercises, evaluate each polynomial for the given value.

606. Evaluate $3y^2 - y + 1$ when:

ⓐ $y = 5$ ⓑ $y = -1$

ⓒ $y = 0$

607. Evaluate $10 - 12x$ when:

ⓐ $x = 3$ ⓑ $x = 0$

ⓒ $x = -1$

608. Randee drops a stone off the 200 foot high cliff into the ocean. The polynomial $-16t^2 + 200$ gives the height of a stone t seconds after it is dropped from the cliff. Find the height after $t = 3$ seconds.

609. A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of *p* dollars each is given by the polynomial $-4p^2 + 460p$. Find the revenue received when $p = 75$ dollars.

6.2 Section 6.2 Use Multiplication Properties of Exponents

Simplify Expressions with Exponents

In the following exercises, simplify.

610. 10^4

611. 17^1

612. $\left(\frac{2}{9}\right)^2$

613. $(0.5)^3$

614. $(-2)^6$

615. -2^6

Simplify Expressions Using the Product Property for Exponents

In the following exercises, simplify each expression.

616. $x^4 \cdot x^3$

617. $p^{15} \cdot p^{16}$

618. $4^{10} \cdot 4^6$

619. $8 \cdot 8^5$

620. $n \cdot n^2 \cdot n^4$

621. $y^c \cdot y^3$

Simplify Expressions Using the Power Property for Exponents

In the following exercises, simplify each expression.

622. $(m^3)^5$

623. $(5^3)^2$

624. $\left(y^4\right)^x$

625. $(3^r)^s$

Simplify Expressions Using the Product to a Power Property

In the following exercises, simplify each expression.

626. $(4a)^2$

627. $(-5y)^3$

628. $(2mn)^5$

629. $(10xyz)^3$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify each expression.

630. $\left(p^2\right)^5 \cdot \left(p^3\right)^6$

631. $(4a^3b^2)^3$

632. $(5x)^2(7x)$

633. $(2q^3)^4(3q)^2$

634. $\left(\frac{1}{3}x^2\right)^2\left(\frac{1}{2}x\right)^3$

635. $\left(\frac{2}{5}m^2n\right)^3$

Multiply Monomials

In the following exercises 8, multiply the monomials.

636. $\left(-15x^2\right)\left(6x^4\right)$

637. $(-9n^7)(-16n)$

638. $(7p^5q^3)(8pq^9)$

639. $\left(\frac{5}{9}ab^2\right)\left(27ab^3\right)$

6.3 Section 6.3 Multiply Polynomials

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

640. $7(a+9)$

641. $-4(y+13)$

642. $-5(r-2)$

643. $p(p+3)$

644. $-m(m+15)$

645. $-6u(2u+7)$

646. $9(b^2+6b+8)$

647. $3q^2(q^2-7q+6)$ 3

648. $(5z-1)z$

649. $(b-4)\cdot 11$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using: ⓐ the Distributive Property, ⓑ the FOIL method, ⓒ the Vertical Method.

650. $(x-4)(x+10)$

651. $(6y-7)(2y-5)$

In the following exercises, multiply the binomials. Use any method.

652. $(x+3)(x+9)$

653. $(y-4)(y-8)$

654. $(p-7)(p+4)$

655. $(q+16)(q-3)$

656. $(5m-8)(12m+1)$

657. $(u^2+6)(u^2-5)$

658. $(9x-y)(6x-5)$

659. $(8mn+3)(2mn-1)$

Multiply a Trinomial by a Binomial

In the following exercises, multiply using ⓐ the Distributive Property, ⓑ the Vertical Method.

660. $(n+1)\left(n^2+5n-2\right)$

661. $(3x-4)\left(6x^2+x-10\right)$

In the following exercises, multiply. Use either method.

662. $(y-2)(y^2-8y+9)$

663. $(7m+1)(m^2-10m-3)$

6.4 Section 6.4 Special Products

Square a Binomial Using the Binomial Squares Pattern

In the following exercises, square each binomial using the Binomial Squares Pattern.

664. $(c+11)^2$

665. $(q-15)^2$

666. $\left(x+\frac{1}{3}\right)^2$

667. $(8u+1)^2$

668. $(3n^3-2)^2$

669. $(4a-3b)^2$

Multiply Conjugates Using the Product of Conjugates Pattern

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

670. $(s-7)(s+7)$

671. $\left(y+\frac{2}{5}\right)\left(y-\frac{2}{5}\right)$

672. $(12c+13)(12c-13)$

673. $(6-r)(6+r)$

674. $\left(u+\frac{3}{4}v\right)\left(u-\frac{3}{4}v\right)$

675. $(5p^4-4q^3)(5p^4+4q^3)$

Recognize and Use the Appropriate Special Product Pattern

In the following exercises, find each product.

676. $(3m+10)^2$

677. $(6a+11)(6a-11)$

678. $(5x+y)(x-5y)$

679. $(c^4+9d)^2$

680. $(p^5+q^5)(p^5-q^5)$

681. $(a^2+4b)(4a-b^2)$

6.5 Section 6.5 Divide Monomials

Simplify Expressions Using the Quotient Property for Exponents

In the following exercises, simplify.

682. $\frac{u^{24}}{u^6}$

683. $\frac{10^{25}}{10^5}$

684. $\frac{3^4}{3^6}$

685. $\frac{v^{12}}{v^{48}}$

686. $\frac{x}{x^5}$

687. $\frac{5}{5^8}$

Simplify Expressions with Zero Exponents

In the following exercises, simplify.

688. 75^0

689. x^0

690. -12^0

691. $\left(-12^0\right)(-12)^0$

692. $25x^0$

693. $(25x)^0$

694. $19n^0-25m^0$

695. $(19n)^0-(25m)^0$

Simplify Expressions Using the Quotient to a Power Property

In the following exercises, simplify.

696. $\left(\frac{2}{5}\right)^3$

697. $\left(\frac{m}{3}\right)^4$

698. $\left(\frac{r}{s}\right)^8$

699. $\left(\frac{x}{2y}\right)^6$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify.

700. $\frac{\left(x^3\right)^5}{x^9}$

701. $\frac{n^{10}}{\left(n^5\right)^2}$

702. $\left(\frac{q^6}{q^8}\right)^3$

703. $\left(\frac{r^8}{r^3}\right)^4$

704. $\left(\frac{c^2}{d^5}\right)^9$

705. $\left(\frac{3x^4}{2y^2}\right)^5$

706. $\left(\frac{v^3 v^9}{v^6}\right)^4$

707. $\frac{\left(3n^2\right)^4\left(-5n^4\right)^3}{\left(-2n^5\right)^2}$

Divide Monomials

In the following exercises, divide the monomials.

708. $-65y^{14} \div 5y^2$

709. $\frac{64a^5 b^9}{-16a^{10} b^3}$

710. $\frac{144x^{15} y^8 z^3}{18x^{10} y^2 z^{12}}$

711. $\frac{\left(8p^6 q^2\right)\left(9p^3 q^5\right)}{16p^8 q^7}$

6.6 Section 6.6 Divide Polynomials

Divide a Polynomial by a Monomial

In the following exercises, divide each polynomial by the monomial.

712. $\frac{42z^2 - 18z}{6}$

713. $\left(35x^2 - 75x\right) \div 5x$

714. $\frac{81n^4 + 105n^2}{-3}$

715. $\frac{550p^6 - 300p^4}{10p^3}$

716. $\left(63xy^3 + 56x^2 y^4\right) \div (7xy)$

717. $\frac{96a^5 b^2 - 48a^4 b^3 - 56a^2 b^4}{8ab^2}$

718. $\frac{57m^2 - 12m + 1}{-3m}$

719. $\frac{105y^5 + 50y^3 - 5y}{5y^3}$

Divide a Polynomial by a Binomial

In the following exercises, divide each polynomial by the binomial.

720. $\left(k^2 - 2k - 99\right) \div (k + 9)$

721. $\left(v^2 - 16v + 64\right) \div (v - 8)$

722. $\left(3x^2 - 8x - 35\right) \div (x - 5)$

723. $\left(n^2 - 3n - 14\right) \div (n + 3)$

724. $\left(4m^3 + m - 5\right) \div (m - 1)$

725. $\left(u^3 - 8\right) \div (u - 2)$

6.7 Section 6.7 Integer Exponents and Scientific Notation

Use the Definition of a Negative Exponent

In the following exercises, simplify.

726. 9^{-2}

727. $(-5)^{-3}$

728. $3 \cdot 4^{-3}$

729. $(6u)^{-3}$

730. $\left(\frac{2}{5}\right)^{-1}$

731. $\left(\frac{3}{4}\right)^{-2}$

Simplify Expressions with Integer Exponents

In the following exercises, simplify.

732. $p^{-2} \cdot p^{8}$

733. $q^{-6} \cdot q^{-5}$

734. $\left(c^{-2} d\right)\left(c^{-3} d^{-2}\right)$

735. $\left(y^{8}\right)^{-1}$

736. $\left(q^{-4}\right)^{-3}$

737. $\dfrac{a^{8}}{a^{12}}$

738. $\dfrac{n^{5}}{n^{-4}}$

739. $\dfrac{r^{-2}}{r^{-3}}$

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

740. 8,500,000

741. 0.00429

742. The thickness of a dime is about 0.053 inches.

743. In 2015, the population of the world was about 7,200,000,000 people.

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

744. 3.8×10^{5}

745. 1.5×10^{10}

746. 9.1×10^{-7}

747. 5.5×10^{-1}

Multiply and Divide Using Scientific Notation

In the following exercises, multiply and write your answer in decimal form.

748. $\left(2 \times 10^{5}\right)\left(4 \times 10^{-3}\right)$

749. $\left(3.5 \times 10^{-2}\right)\left(6.2 \times 10^{-1}\right)$

In the following exercises, divide and write your answer in decimal form.

750. $\dfrac{8 \times 10^{5}}{4 \times 10^{-1}}$

751. $\dfrac{9 \times 10^{-5}}{3 \times 10^{2}}$

PRACTICE TEST

752. For the polynomial $10x^4 + 9y^2 - 1$

ⓐ Is it a monomial, binomial, or trinomial?

ⓑ What is its degree?

In the following exercises, simplify each expression.

753. $\left(12a^2 - 7a + 4\right) + \left(3a^2 + 8a - 10\right)$

754. $\left(9p^2 - 5p + 1\right) - \left(2p^2 - 6\right)$

755. $\left(-\frac{2}{5}\right)^3$

756. $u \cdot u^4$

757. $\left(4a^3 b^5\right)^2$

758. $\left(-9r^4 s^5\right)\left(4rs^7\right)$

759. $3k\left(k^2 - 7k + 13\right)$

760. $(m + 6)(m + 12)$

761. $(v - 9)(9v - 5)$

762. $(4c - 11)(3c - 8)$

763. $(n - 6)\left(n^2 - 5n + 4\right)$

764. $(2x - 15y)(5x + 7y)$

765. $(7p - 5)(7p + 5)$

766. $(9v - 2)^2$

767. $\frac{3^8}{3^{10}}$

768. $\left(\frac{m^4 \cdot m}{m^3}\right)^6$

769. $\left(87x^{15} y^3 z^{22}\right)^0$

770. $\frac{80c^8 d^2}{16cd^{10}}$

771. $\frac{12x^2 + 42x - 6}{2x}$

772. $\left(70xy^4 + 95x^3 y\right) \div 5xy$

773. $\frac{64x^3 - 1}{4x - 1}$

774. $\left(y^2 - 5y - 18\right) \div (y + 3)$

775. 5^{-2}

776. $(4m)^{-3}$

777. $q^{-4} \cdot q^{-5}$

778. $\frac{n^{-2}}{n^{-10}}$

779. Convert 83,000,000 to scientific notation.

780. Convert 6.91×10^{-5} to decimal form.

In the following exercises, simplify, and write your answer in decimal form.

781. $\left(3.4 \times 10^9\right)\left(2.2 \times 10^{-5}\right)$

782. $\frac{8.4 \times 10^{-3}}{4 \times 10^3}$

783. A helicopter flying at an altitude of 1000 feet drops a rescue package. The polynomial $-16t^2 + 1000$ gives the height of the package t seconds a after it was dropped. Find the height when $t = 6$ seconds.

7 FACTORING

Figure 7.1 The Sydney Harbor Bridge is one of Australia's most photographed landmarks. It is the world's largest steel arch bridge with the top of the bridge standing 134 meters above the harbor. Can you see why it is known by the locals as the "Coathanger"?

Chapter Outline

Introduction

Quadratic expressions may be used to model physical properties of a large bridge, the trajectory of a baseball or rocket, and revenue and profit of a business. By factoring these expressions, specific characteristics of the model can be identified. In this chapter, you will explore the process of factoring expressions and see how factoring is used to solve certain types of equations.

7.1 Greatest Common Factor and Factor by Grouping

Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

Be Prepared!

Before you get started, take this readiness quiz.

1. Factor 56 into primes.
   If you missed this problem, review **Example 1.7**.
2. Find the least common multiple of 18 and 24.
   If you missed this problem, review **Example 1.10**.
3. Simplify $-3(6a+11)$.
   If you missed this problem, review **Example 1.135**.

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will be reversing this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.

multiply

$$\underbrace{8 \cdot 7}_{factors} = \underbrace{56}_{product} \qquad \underbrace{2x(x+3)}_{factors} = \underbrace{2x^2 + 6x}_{product}$$

factor

We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

Greatest Common Factor

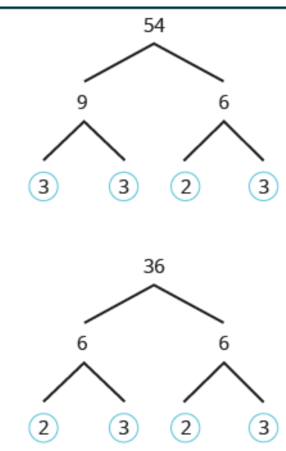
The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

First we'll find the GCF of two numbers.

EXAMPLE 7.1 HOW TO FIND THE GREATEST COMMON FACTOR OF TWO OR MORE EXPRESSIONS

Find the GCF of 54 and 36.

Solution

Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.	Factor 54 and 36.	
Step 2. In each column, circle the common factors.	Circle the 2, 3, and 3 that are shared by both numbers.	$36 = 2 \cdot 2 \cdot 3 \cdot 3$ $18 = 2 \cdot \quad 3 \cdot 3 \cdot 3$
Step 3. Bring down the common factors that all expressions share.	Bring down the 2, 3, and 3, and then multiply.	$\text{GCF} = 2 \cdot \quad 3 \cdot 3$
Step 4. Multiply the factors.		$\text{GCF} = 18$ The GCF of 54 and 36 is 18.

Notice that, because the GCF is a factor of both numbers, 54 and 36 can be written as multiples of 18.

$$54 = 18 \cdot 3$$
$$36 = 18 \cdot 2$$

TRY IT : : 7.1 Find the GCF of 48 and 80.

TRY IT : : 7.2 Find the GCF of 18 and 40.

We summarize the steps we use to find the GCF below.

HOW TO : : FIND THE GREATEST COMMON FACTOR (GCF) OF TWO EXPRESSIONS.

Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.

Step 2. List all factors—matching common factors in a column. In each column, circle the common factors.

Step 3. Bring down the common factors that all expressions share.

Step 4. Multiply the factors.

In the first example, the GCF was a constant. In the next two examples, we will get variables in the greatest common factor.

EXAMPLE 7.2

Find the greatest common factor of $27x^3$ and $18x^4$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	$27x^3 = 3 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x$ $18x^4 = 2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x \cdot x$
Bring down the common factors.	$\text{GCF} = 3 \cdot 3 \cdot x \cdot x \cdot x$
Multiply the factors.	$\text{GCF} = 9x^3$
	The GCF of $27x^3$ and $18x^4$ is $9x^3$.

TRY IT : : 7.3 Find the GCF: $12x^2$, $18x^3$.

TRY IT : : 7.4 Find the GCF: $16y^2$, $24y^3$.

EXAMPLE 7.3

Find the GCF of $4x^2y$, $6xy^3$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	$4x^2y = 2 \cdot 2 \cdot x \cdot x \cdot y$ $6xy^3 = 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y$
Bring down the common factors.	$\text{GCF} = 2 \cdot x \cdot y$
Multiply the factors.	$\text{GCF} = 2xy$

	The GCF of $4x^2y$ and $6xy^3$ is $2xy$.

> **TRY IT : : 7.5** Find the GCF: $6ab^4,\ 8a^2b$.

> **TRY IT : : 7.6** Find the GCF: $9m^5n^2,\ 12m^3n$.

EXAMPLE 7.4

Find the GCF of: $21x^3,\ 9x^2,\ 15x$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	$21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$
Bring down the common factors.	$\text{GCF} = 3 \cdot x$
Multiply the factors.	$\text{GCF} = 3x$
	The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

> **TRY IT : : 7.7** Find the greatest common factor: $25m^4,\ 35m^3,\ 20m^2$.

> **TRY IT : : 7.8** Find the greatest common factor: $14x^3,\ 70x^2,\ 105x$.

Factor the Greatest Common Factor from a Polynomial

Just like in arithmetic, where it is sometimes useful to represent a number in factored form (for example, 12 as $2 \cdot 6 \text{ or } 3 \cdot 4$), in algebra, it can be useful to represent a polynomial in factored form. One way to do this is by finding the GCF of all the terms. Remember, we multiply a polynomial by a monomial as follows:

$$\begin{array}{cl} 2(x+7) & \text{factors} \\ 2 \cdot x + 2 \cdot 7 & \\ 2x + 14 & \text{product} \end{array}$$

Now we will start with a product, like $2x + 14$, and end with its factors, $2(x + 7)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

Distributive Property

If a, b, c are real numbers, then

$$a(b+c) = ab + ac \qquad \text{and} \qquad ab + ac = a(b+c)$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

EXAMPLE 7.5 HOW TO FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL

Factor: $4x + 12$.

Solution

Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $4x$ and 12.	$4x = 2 \cdot 2 \cdot \quad \cdot x$ $12 = 2 \cdot 2 \cdot 3$ GCF $= 2 \cdot 2$ GCF $= 4$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $4x$ and 12 as products of their GCF, 4. $4x = 4 \cdot x$ $12 = 4 \cdot 3$	$4x + 12$ $4 \cdot x + 4 \cdot 3$
Step 3. Use the "reverse" Distributive Property to factor the expression.		$4(x + 3)$
Step 4. Check by multiplying the factors.		$4(x + 3)$ $4 \cdot x + 4 \cdot 3$ $4x + 12$ ✓

TRY IT : : 7.9 Factor: $6a + 24$.

TRY IT : : 7.10 Factor: $2b + 14$.

HOW TO : : FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL.

Step 1. Find the GCF of all the terms of the polynomial.
Step 2. Rewrite each term as a product using the GCF.
Step 3. Use the "reverse" Distributive Property to factor the expression.
Step 4. Check by multiplying the factors.

Factor as a Noun and a Verb

We use "factor" as both a noun and a verb.

Noun	7 is a factor of 14
Verb	factor 3 from $3a + 3$

EXAMPLE 7.6

Factor: $5a + 5$.

Solution

Find the GCF of $5a$ and 5.	$5a = 5 \cdot a$ $5 = 5$ $\text{GCF} = 5$
	$5a + 5$
Rewrite each term as a product using the GCF.	$5 \cdot a + 5 \cdot 1$
Use the Distributive Property "in reverse" to factor the GCF.	$5(a + 1)$
Check by mulitplying the factors to get the orginal polynomial.	
$5(a + 1)$	
$5 \cdot a + 5 \cdot 1$	
$5a + 5$ ✓	

TRY IT : : 7.11 Factor: $14x + 14$.

TRY IT : : 7.12 Factor: $12p + 12$.

The expressions in the next example have several factors in common. Remember to write the GCF as the product of all the common factors.

EXAMPLE 7.7

Factor: $12x - 60$.

Solution

Find the GCF of $12x$ and 60.	$12x = 2 \cdot 2 \cdot 3 \cdot x$ $60 = 2 \cdot 2 \cdot 3 \cdot 5$ $\text{GCF} = 2 \cdot 2 \cdot 3$ $\text{GCF} = 12$
	$12x - 60$
Rewrite each term as a product using the GCF.	$12 \cdot x - 12 \cdot 5$
Factor the GCF.	$12(x - 5)$
Check by mulitplying the factors.	
$12(x - 5)$	
$12 \cdot x - 12 \cdot 5$	
$12x - 60$ ✓	

TRY IT : : 7.13 Factor: $18u - 36$.

TRY IT : : 7.14 Factor: $30y - 60$.

Now we'll factor the greatest common factor from a trinomial. We start by finding the GCF of all three terms.

EXAMPLE 7.8

Factor: $4y^2 + 24y + 28$.

Solution

We start by finding the GCF of all three terms.

Find the GCF of $4y^2$, $24y$ and 28.	$4y^2 = 2 \cdot 2 \cdot y \cdot y$ $24y = 2 \cdot 2 \cdot 2 \cdot 3 \cdot y$ $28 = 2 \cdot 2 \cdot 7$ $\text{GCF} = 2 \cdot 2$ $\text{GCF} = 4$
	$4y^2 + 24y + 28$
Rewrite each term as a product using the GCF.	$4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$
Factor the GCF.	$4(y^2 + 6y + 7)$
Check by mulitplying.	
$4(y^2 + 6y + 7)$	
$4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$	
$4y^2 + 24y + 28$ ✓	

TRY IT : : 7.15 Factor: $5x^2 - 25x + 15$.

TRY IT : : 7.16 Factor: $3y^2 - 12y + 27$.

EXAMPLE 7.9

Factor: $5x^3 - 25x^2$.

Solution

Find the GCF of $5x^3$ and $25x^2$.	$5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ $\text{GCF} = 5 \cdot x \cdot x$ $\text{GCF} = 5x^2$
	$5x^3 - 25x^2$
Rewrite each term.	$5x^2 \cdot x - 5x^2 \cdot 5$

Factor the GCF.	$5x^2(x-5)$
Check.	
$5x^2(x-5)$	
$5x^2 \cdot x - 5x^2 \cdot 5$	
$5x^3 - 25x^2$ ✓	

TRY IT : : 7.17 Factor: $2x^3 + 12x^2$.

TRY IT : : 7.18 Factor: $6y^3 - 15y^2$.

EXAMPLE 7.10

Factor: $21x^3 - 9x^2 + 15x$.

Solution

In a previous example we found the GCF of $21x^3, 9x^2, 15x$ to be $3x$.

	$21x^3 - 9x^2 + 15x$
Rewrite each term using the GCF, $3x$.	$3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5$
Factor the GCF.	$3x(7x^2 - 3x + 5)$
Check.	
$3x(7x^2 - 3x + 5)$	
$3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5$	
$21x^3 - 9x^2 + 15x$ ✓	

TRY IT : : 7.19 Factor: $20x^3 - 10x^2 + 14x$.

TRY IT : : 7.20 Factor: $24y^3 - 12y^2 - 20y$.

EXAMPLE 7.11

Factor: $8m^3 - 12m^2n + 20mn^2$.

Solution

Find the GCF of $8m^3$, $12m^2n$, $20mn^2$.	$8m^3 = 2 \cdot 2 \cdot 2 \quad m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$ $\text{GCF} = 2 \cdot 2 \cdot m$ $\text{GCF} = 4m$
	$8m^3 - 12m^2n + 20mn^2$
Rewrite each term.	$4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$
Factor the GCF.	$4m(2m^2 - 3mn + 5n^2)$
Check.	
$4m(2m^2 - 3mn + 5n^2)$	
$4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$	
$8m^3 - 12m^2n + 20mn^2$ ✓	

TRY IT : : 7.21 Factor: $9xy^2 + 6x^2y^2 + 21y^3$.

TRY IT : : 7.22 Factor: $3p^3 - 6p^2q + 9pq^3$.

When the leading coefficient is negative, we factor the negative out as part of the GCF.

EXAMPLE 7.12

Factor: $-8y - 24$.

Solution

When the leading coefficient is negative, the GCF will be negative.

Ignoring the signs of the terms, we first find the GCF of 8*y* and 24 is 8. Since the expression −8*y* − 24 has a negative leading coefficient, we use −8 as the GCF.	$8y = 2 \cdot 2 \cdot 2 \cdot y$ $24 = 2 \cdot 2 \cdot 2 \cdot 3$ $\text{GCF} = 2 \cdot 2 \cdot 2$ $\text{GCF} = 8$
Rewrite each term using the GCF.	$-8y - 24$ $-8 \cdot y + (-8) \cdot 3$
Factor the GCF.	$-8(y + 3)$
Check.	
$-8(y + 3)$	
$-8 \cdot y + (-8) \cdot 3$	

$-8y - 24$ ✓

TRY IT : : 7.23 Factor: $-16z - 64$.

TRY IT : : 7.24 Factor: $-9y - 27$.

EXAMPLE 7.13

Factor: $-6a^2 + 36a$.

Solution

The leading coefficient is negative, so the GCF will be negative.?

Since the leading coefficient is negative, the GCF is negative, $-6a$.	$6a^2 = 2 \cdot 3 \cdot a \cdot a$ $36a = 2 \cdot 2 \cdot 3 \cdot 3 \cdot a$ $\text{GCF} = 2 \cdot 3 \cdot a$ $\text{GCF} = 6a$ $-6a^2 + 36a$
Rewrite each term using the GCF.	$-6a \cdot a - (-6a) \cdot 6$
Factor the GCF.	$-6a(a - 6)$
Check.	
$-6a(a - 6)$	
$-6a \cdot a + (-6a)(-6)$	
$-6a^2 + 36a$ ✓	

TRY IT : : 7.25 Factor: $-4b^2 + 16b$.

TRY IT : : 7.26 Factor: $-7a^2 + 21a$.

EXAMPLE 7.14

Factor: $5q(q + 7) - 6(q + 7)$.

Solution

The GCF is the binomial $q + 7$.

	$5q(q + 7) - 6(q + 7)$
Factor the GCF, $(q + 7)$.	$(q + 7)(5q - 6)$
Check on your own by multiplying.	

TRY IT : : 7.27 Factor: $4m(m+3)-7(m+3)$.

TRY IT : : 7.28 Factor: $8n(n-4)+5(n-4)$.

Factor by Grouping

When there is no common factor of all the terms of a polynomial, look for a common factor in just some of the terms. When there are four terms, a good way to start is by separating the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find a common factor emerges from both parts.

(Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.)

EXAMPLE 7.15 HOW TO FACTOR BY GROUPING

Factor: $xy+3y+2x+6$.

Solution

Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms? No, so let's separate the first two terms from the second two.	$xy+3y+2x+6$ $\underbrace{xy+3y}+\underbrace{2x+6}$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms. Factor the GCF from the second two terms.	$y(x+3)+\underbrace{2x+6}$ $y(x+3)+2(x+3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x+3)$. Factor out the common factor.	$y(x+3)+2(x+3)$ $(x+3)(y+2)$
Step 4. Check.	Multiply $(x+3)(y+2)$. Is the product the original expression?	$(x+3)(y+2)$ $xy+2x+3y+6$ $xy+3y+2x+6$ ✓

TRY IT : : 7.29 Factor: $xy+8y+3x+24$.

TRY IT : : 7.30 Factor: $ab+7b+8a+56$.

HOW TO : : FACTOR BY GROUPING.

Step 1. Group terms with common factors.
Step 2. Factor out the common factor in each group.
Step 3. Factor the common factor from the expression.
Step 4. Check by multiplying the factors.

EXAMPLE 7.16

Factor: $x^2+3x-2x-6$.

Solution

There is no GCF in all four terms.	$x^2 + 3x \;\; -2x - 6$
Separate into two parts.	$\underbrace{x^2 + 3x}\;\underbrace{-2x - 6}$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$x(x+3) - 2(x+3)$ $(x+3)(x-2)$

Check on your own by multiplying.

TRY IT : : 7.31 Factor: $x^2 + 2x - 5x - 10$.

TRY IT : : 7.32 Factor: $y^2 + 4y - 7y - 28$.

MEDIA : :

Access these online resources for additional instruction and practice with greatest common factors (GFCs) and factoring by grouping.

- **Greatest Common Factor (GCF) (https://openstax.org/l/25GCF1)**
- **Factoring Out the GCF of a Binomial (https://openstax.org/l/25GCF2)**
- **Greatest Common Factor (GCF) of Polynomials (https://openstax.org/l/25GCF3)**

7.1 EXERCISES

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

1. 8, 18

2. 24, 40

3. 72, 162

4. 150, 275

5. 10*a*, 50

6. 5*b*, 30

7. $3x,\ 10x^2$

8. $21b^2,\ 14b$

9. $8w^2,\ 24w^3$

10. $30x^2,\ 18x^3$

11. $10p^3q,\ 12pq^2$

12. $8a^2b^3,\ 10ab^2$

13. $12m^2n^3,\ 30m^5n^3$

14. $28x^2y^4,\ 42x^4y^4$

15. $10a^3,\ 12a^2,\ 14a$

16. $20y^3,\ 28y^2,\ 40y$

17. $35x^3,\ 10x^4,\ 5x^5$

18. $27p^2,\ 45p^3,\ 9p^4$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

19. $4x + 20$

20. $8y + 16$

21. $6m + 9$

22. $14p + 35$

23. $9q + 9$

24. $7r + 7$

25. $8m - 8$

26. $4n - 4$

27. $9n - 63$

28. $45b - 18$

29. $3x^2 + 6x - 9$

30. $4y^2 + 8y - 4$

31. $8p^2 + 4p + 2$

32. $10q^2 + 14q + 20$

33. $8y^3 + 16y^2$

34. $12x^3 - 10x$

35. $5x^3 - 15x^2 + 20x$

36. $8m^2 - 40m + 16$

37. $12xy^2 + 18x^2y^2 - 30y^3$

38. $21pq^2 + 35p^2q^2 - 28q^3$

39. $-2x - 4$

40. $-3b + 12$

41. $5x(x + 1) + 3(x + 1)$

42. $2x(x - 1) + 9(x - 1)$

43. $3b(b - 2) - 13(b - 2)$

44. $6m(m - 5) - 7(m - 5)$

Factor by Grouping

In the following exercises, factor by grouping.

45. $xy + 2y + 3x + 6$

46. $mn + 4n + 6m + 24$

47. $uv - 9u + 2v - 18$

48. $pq - 10p + 8q - 80$

49. $b^2 + 5b - 4b - 20$

50. $m^2 + 6m - 12m - 72$

51. $p^2 + 4p - 9p - 36$

52. $x^2 + 5x - 3x - 15$

Mixed Practice

In the following exercises, factor.

53. $-20x-10$

54. $5x^3-x^2+x$

55. $3x^3-7x^2+6x-14$

56. x^3+x^2-x-1

57. $x^2+xy+5x+5y$

58. $5x^3-3x^2-5x-3$

Everyday Math

59. Area of a rectangle The area of a rectangle with length 6 less than the width is given by the expression w^2-6w, where $w=$ width. Factor the greatest common factor from the polynomial.

60. Height of a baseball The height of a baseball t seconds after it is hit is given by the expression $-16t^2+80t+4$. Factor the greatest common factor from the polynomial.

Writing Exercises

61. The greatest common factor of 36 and 60 is 12. Explain what this means.

62. What is the GCF of $y^4, y^5, \text{and } y^{10}$? Write a general rule that tells you how to find the GCF of $y^a, y^b, \text{and} y^c$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
find the greatest common factor of two or more expressions.			
factor the greatest common factor from a polynomial.			
factor by grouping.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!*

...with some help. *This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential—every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.*

7.2 Factor Quadratic Trinomials with Leading Coefficient 1

Learning Objectives

By the end of this section, you will be able to:

- Factor trinomials of the form $x^2 + bx + c$
- Factor trinomials of the form $x^2 + bxy + cy^2$

Be Prepared!

Before you get started, take this readiness quiz.

1. Multiply: $(x + 4)(x + 5)$.
   If you missed this problem, review **Example 6.38**.
2. Simplify: ⓐ $-9 + (-6)$ ⓑ $-9 + 6$.
   If you missed this problem, review **Example 1.37**.
3. Simplify: ⓐ $-9(6)$ ⓑ $-9(-6)$.
   If you missed this problem, review **Example 1.46**.
4. Simplify: ⓐ $|-5|$ ⓑ $|3|$.
   If you missed this problem, review **Example 1.33**.

Factor Trinomials of the Form $x^2 + bx + c$

You have already learned how to multiply binomials using FOIL. Now you'll need to "undo" this multiplication—to start with the product and end up with the factors. Let's look at an example of multiplying binomials to refresh your memory.

$$(x + 2)(x + 3) \quad \text{factors}$$
$$F \quad O \quad I \quad L$$
$$x^2 + 3x + 2x + 6$$
$$x^2 + 5x + 6 \quad \text{product}$$

To factor the trinomial means to start with the prod uct, $x^2 + 5x + 6$, and end with the factors, $(x + 2)(x + 3)$. You need to think about where each of the terms in the trinomial came from.

The *first term* came from multiplying the first term in each binomial. So to get x^2 in the product, each binomial must start with an *x*.

$$\begin{array}{c} x^2 + 5x + 6 \\ (x \quad)(x \quad) \end{array}$$

The *last term* in the trinomial came from multiplying the last term in each binomial. So the last terms must multiply to 6.

What two numbers multiply to 6?

The factors of 6 could be 1 and 6, or 2 and 3. How do you know which pair to use?

Consider the *middle term*. It came from adding the outer and inner terms.

So the numbers that must have a product of 6 will need a sum of 5. We'll test both possibilities and summarize the results in **Table 7.1**—the table will be very helpful when you work with numbers that can be factored in many different ways.

Factors of 6	Sum of factors
$1, 6$	$1 + 6 = 7$
$2, 3$	$2 + 3 = 5$

Table 7.1

We see that 2 and 3 are the numbers that multiply to 6 and add to 5. So we have the factors of x^2+5x+6. They are $(x+2)(x+3)$.

$$\begin{array}{ll} x^2+5x+6 & \text{product} \\ (x+2)(x+3) & \text{factors} \end{array}$$

You should check this by multiplying.

Looking back, we started with x^2+5x+6, which is of the form x^2+bx+c, where $b=5$ and $c=6$. We factored it into two binomials of the form $(x+m)$ and $(x+n)$.

$$\begin{array}{ll} x^2+5x+6 & x^2+bx+c \\ (x+2)(x+3) & (x+m)(x+n) \end{array}$$

To get the correct factors, we found two numbers *m* and *n* whose product is *c* and sum is *b*.

EXAMPLE 7.17 HOW TO FACTOR TRINOMIALS OF THE FORM x^2+bx+c

Factor: $x^2+7x+12$.

Solution

Step 1. Write the factors as two binomials with first terms *x*.	Write two sets of parentheses and put *x* as the first term.	$x^2+7x+12$ $(x\quad)(x\quad)$
Step 2. Find two numbers *m and n* that multiply to *c*, $m \cdot n = c$ add to *b*, $m+n=b$	Find two numbers that multiply to 12 and add to 7. **Factors of 12** / **Sum of factors** 1, 12 / $1+12=13$ 2, 6 / $2+6=8$ 3, 4 / $3+4=7$*	
Step 3. Use *m and n* as the last terms of the factors.	Use 3 and 4 as the last terms of the binomials.	$(x+3)(x+4)$
Step 4. Check by multiplying the factors.		$(x+3)(x+4)$ $x^2+4x+3x+12$ $x^2+7x+12$ ✓

TRY IT : : 7.33 Factor: x^2+6x+8.

TRY IT : : 7.34 Factor: $y^2+8y+15$.

Let's summarize the steps we used to find the factors.

HOW TO : : FACTOR TRINOMIALS OF THE FORM x^2+bx+c.

Step 1. Write the factors as two binomials with first terms *x*: $(x\quad)(x\quad)$.

Step 2. Find two numbers *m* and *n* that
Multiply to *c*, $m \cdot n = c$
Add to *b*, $m+n=b$

Step 3. Use *m* and *n* as the last terms of the factors: $(x+m)(x+n)$.

Step 4. Check by multiplying the factors.

EXAMPLE 7.18

Factor: $u^2 + 11u + 24$.

Solution

Notice that the variable is *u*, so the factors will have first terms *u*.

	$u^2 + 11u + 24$
Write the factors as two binomials with fir t terms u.	$(u \quad)(u \quad)$

Find two numbers that: multiply to 24 and add to 11.

Factors of 24	Sum of factors
$1, 24$	$1 + 24 = 25$
$2, 12$	$2 + 12 = 14$
$3, 8$	$3 + 8 = 11^*$
$4, 6$	$4 + 6 = 10$

Use 3 and 8 as the last terms of the binomials.	$(u + 3)(u + 8)$

Check.

$$(u + 3)(u + 8)$$
$$u^2 + 3u + 8u + 24$$
$$u^2 + 11u + 24 \checkmark$$

TRY IT : : 7.35 Factor: $q^2 + 10q + 24$.

TRY IT : : 7.36 Factor: $t^2 + 14t + 24$.

EXAMPLE 7.19

Factor: $y^2 + 17y + 60$.

Solution

	$y^2 + 17y + 60$
Write the factors as two binomials with fir t terms y.	$(y \quad)(y \quad)$

Find two numbers that multiply to 60 and add to 17.

Factors of 60	Sum of factors
$1, 60$	$1 + 60 = 61$
$2, 30$	$2 + 30 = 32$
$3, 20$	$3 + 20 = 23$
$4, 15$	$4 + 15 = 19$
$5, 12$	$5 + 12 = 17^{*}$
$6, 10$	$6 + 10 = 16$

Use 5 and 12 as the last terms. $(y + 5)(y + 12)$

Check.

$$(y + 5)(y + 12)$$
$$(y^2 + 12y + 5y + 60)$$
$$(y^2 + 17y + 60) \checkmark$$

TRY IT : : 7.37 Factor: $x^2 + 19x + 60$.

TRY IT : : 7.38 Factor: $v^2 + 23v + 60$.

Factor Trinomials of the Form $x^2 + bx + c$ with b Negative, c Positive

In the examples so far, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

Remember: To get a negative sum and a positive product, the numbers must both be negative.

Again, think about FOIL and where each term in the trinomial came from. Just as before,

- the first term, x^2, comes from the product of the two first terms in each binomial factor, *x* and *y*;
- the positive last term is the product of the two last terms
- the negative middle term is the sum of the outer and inner terms.

How do you get a *positive product* and a *negative sum*? With two negative numbers.

EXAMPLE 7.20

Factor: $t^2 - 11t + 28$.

Solution

Again, with the positive last term, 28, and the negative middle term, $-11t$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

$$t^2 - 11t + 28$$

Write the factors as two binomials with fir t terms t. $(t \quad)(t \quad)$

Find two numbers that: multiply to 28 and add to -11.

Factors of 28	Sum of factors
$-1,\ -28$	$-1+(-28)=-29$
$-2,\ -14$	$-2+(-14)=-16$
$-4,\ -7$	$-4+(-7)=-11^{*}$

Use $-4,\ -7$ as the last terms of the binomials. $(t-4)(t-7)$

Check.

$(t-4)(t-7)$
$t^2-7t-4t+28$
$t^2-11t+28$ ✓

TRY IT : : 7.39 Factor: $u^2-9u+18$.

TRY IT : : 7.40 Factor: $y^2-16y+63$.

Factor Trinomials of the Form x^2+bx+c with *c* Negative

Now, what if the last term in the trinomial is negative? Think about FOIL. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

Remember: To get a negative product, the numbers must have different signs.

EXAMPLE 7.21

Factor: z^2+4z-5.

Solution

To get a negative last term, multiply one positive and one negative. We need factors of -5 that add to positive 4.

Factors of -5	Sum of factors
$1,\ -5$	$1+(-5)=-4$
$-1,\ 5$	$-1+5=4^{*}$

Notice: We listed both $1,\ -5$ and $-1,\ 5$ to make sure we got the sign of the middle term correct.

	z^2+4z-5
Factors will be two binomials with fir t terms z.	$(z\quad)(z\quad)$
Use $-1,\ 5$ as the last terms of the binomials.	$(z-1)(z+5)$

Check.

$(z-1)(z+5)$
$z^2+5z-1z-5$
z^2+4z-5 ✓

TRY IT : : 7.41 Factor: $h^2 + 4h - 12$.

TRY IT : : 7.42 Factor: $k^2 + k - 20$.

Let's make a minor change to the last trinomial and see what effect it has on the factors.

EXAMPLE 7.22

Factor: $z^2 - 4z - 5$.

Solution

This time, we need factors of -5 that add to -4.

Factors of -5	Sum of factors
$1, -5$	$1 + (-5) = -4^*$
$-1, 5$	$-1 + 5 = 4$

	$z^2 - 4z - 5$
Factors will be two binomials with fir t terms z.	$(z \quad)(z \quad)$
Use $1, -5$ as the last terms of the binomials.	$(z + 1)(z - 5)$
Check.	

$$(z + 1)(z - 5)$$
$$z^2 - 5z + 1z - 5$$
$$z^2 - 4z - 5 \checkmark$$

Notice that the factors of $z^2 - 4z - 5$ are very similar to the factors of $z^2 + 4z - 5$. It is very important to make sure you choose the factor pair that results in the correct sign of the middle term.

TRY IT : : 7.43 Factor: $x^2 - 4x - 12$.

TRY IT : : 7.44 Factor: $y^2 - y - 20$.

EXAMPLE 7.23

Factor: $q^2 - 2q - 15$.

Solution

	$q^2 - 2q - 15$
Factors will be two binomials with fir t terms q.	$(q \quad)(q \quad)$
You can use $3, -5$ as the last terms of the binomials.	$(q + 3)(q - 5)$

Factors of -15	Sum of factors
$1, -15$	$1 + (-15) = -14$
$-1, 15$	$-1 + 15 = 14$
$3, -5$	$3 + (-5) = -2^{*}$
$-3, 5$	$-3 + 5 = 2$

Check.

$$(q + 3)(q - 5)$$

$$q^2 - 5q + 3q - 15$$

$$q^2 - 2q - 15 \checkmark$$

TRY IT : : 7.45 Factor: $r^2 - 3r - 40$.

TRY IT : : 7.46 Factor: $s^2 - 3s - 10$.

Some trinomials are prime. The only way to be certain a trinomial is prime is to list all the possibilities and show that none of them work.

EXAMPLE 7.24

Factor: $y^2 - 6y + 15$.

Solution

	$y^2 - 6y + 15$
Factors will be two binomials with fir t terms y.	$(y \quad)(y \quad)$

Factors of 15	Sum of factors
$-1, -15$	$-1 + (-15) = -16$
$-3, -5$	$-3 + (-5) = -8$

As shown in the table, none of the factors add to -6; therefore, the expression is prime.

TRY IT : : 7.47 Factor: $m^2 + 4m + 18$.

TRY IT : : 7.48 Factor: $n^2 - 10n + 12$.

EXAMPLE 7.25

Factor: $2x + x^2 - 48$.

Solution

	$2x + x^2 - 48$
First we put the terms in decreasing degree order.	$x^2 + 2x - 48$
Factors will be two binomials with fir t terms x.	$(x \quad)(x \quad)$

As shown in the table, you can use $-6,\ 8$ as the last terms of the binomials.

$$(x - 6)(x + 8)$$

Factors of -48	Sum of factors
$-1,\ 48$	$-1 + 48 = 47$
$-2,\ 24$ $-3,\ 16$ $-4,\ 12$ $-6,\ 8$	$-2 + 24 = 22$ $-3 + 16 = 13$ $-4 + 12 = 8$ $-6 + 8 = 2$

Check.

$$(x - 6)(x + 8)$$

$$x^2 - 6q + 8q - 48$$

$$x^2 + 2x - 48 \checkmark$$

TRY IT : : 7.49 Factor: $9m + m^2 + 18$.

TRY IT : : 7.50 Factor: $-7n + 12 + n^2$.

Let's summarize the method we just developed to factor trinomials of the form $x^2 + bx + c$.

HOW TO : : FACTOR TRINOMIALS.

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

$$x^2 + bx + c$$
$$(x + m)(x + n)$$

When *c* is positive, *m* and *n* have the same sign.

b positive	b negative
$m,\ n$ positive	$m,\ n$ negative
$x^2 + 5x + 6$	$x^2 - 6x + 8$
$(x + 2)(x + 3)$	$(x - 4)(x - 2)$
same signs	same signs

When *c* is negative, *m* and *n* have opposite signs.

$x^2 + x - 12$	$x^2 - 2x - 15$
$(x + 4)(x - 3)$	$(x - 5)(x + 3)$
opposite signs	opposite signs

Notice that, in the case when *m* and *n* have opposite signs, the sign of the one with the larger absolute value matches the sign of *b*.

Factor Trinomials of the Form $x^2 + bxy + cy^2$

Sometimes you'll need to factor trinomials of the form $x^2 + bxy + cy^2$ with two variables, such as $x^2 + 12xy + 36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x \cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain *y*. To get the coefficients *b* and *c*, you use the same process summarized in the previous objective.

EXAMPLE 7.26

Factor: $x^2 + 12xy + 36y^2$.

Solution

	$x^2 + 12xy + 36y^2$
Note that the fir t terms are x, last terms contain y.	$(x\_y)(x\_y)$

Find the numbers that multiply to 36 and add to 12.

Factors of 36	Sum of factors
1, 36	$1 + 36 = 37$
2, 18	$2 + 18 = 20$
3, 12	$3 + 12 = 15$
4, 9	$4 + 9 = 13$
6, 6	$6 + 6 = 12^*$

Use 6 and 6 as the coefficients of he last terms. $(x+6y)(x+6y)$

Check your answer.

$$(x+6y)(x+6y)$$
$$x^2+6xy+6xy+36y^2$$
$$x^2+12xy+36y^2 \checkmark$$

TRY IT : : 7.51 Factor: $u^2+11uv+28v^2$.

TRY IT : : 7.52 Factor: $x^2+13xy+42y^2$.

EXAMPLE 7.27

Factor: $r^2-8rx-9s^2$.

Solution

We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$r^2-8rx-9s^2$$

Note that the fir t terms are r, last terms contain s. $(r\_s)(r\_s)$

Find the numbers that multiply to -9 and add to -8.

Factors of -9	Sum of factors
$1, -9$	$1+(-9)=-8$*
$-1, 9$	$-1+9=8$
$3, -3$	$3+(-3)=0$

Use $1, -9$ as coefficients of he last terms. $(r+s)(r-9s)$

Check your answer.

$$(r-9s)(r+s)$$
$$r^2+rs-9rs-9s^2$$
$$r^2-8rs-9s^2 \checkmark$$

TRY IT : : 7.53 Factor: $a^2-11ab+10b^2$.

TRY IT : : 7.54 Factor: $m^2-13mn+12n^2$.

EXAMPLE 7.28

Factor: $u^2-9uv-12v^2$.

Solution

We need u in the first term of each binomial and v in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

	$u^2 - 9uv - 12v^2$
Note that the fir t terms are u, last terms contain v.	$(u\_v)(u\_v)$

Find the numbers that multiply to -12 and add to -9.

Factors of -12	Sum of factors
$1, -12$	$1+(-12)=-11$
$-1, 12$	$-1+12=11$
$2, -6$	$2+(-6)=-4$
$-2, 6$	$-2+6=4$
$3, -4$	$3+(-4)=-1$
$-3, 4$	$-3+4=1$

Note there are no factor pairs that give us -9 as a sum. The trinomial is prime.

TRY IT : : 7.55 Factor: $x^2 - 7xy - 10y^2$.

TRY IT : : 7.56 Factor: $p^2 + 15pq + 20q^2$.

7.2 EXERCISES

Practice Makes Perfect

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

63. $x^2 + 4x + 3$

64. $y^2 + 8y + 7$

65. $m^2 + 12m + 11$

66. $b^2 + 14b + 13$

67. $a^2 + 9a + 20$

68. $m^2 + 7m + 12$

69. $p^2 + 11p + 30$

70. $w^2 + 10x + 21$

71. $n^2 + 19n + 48$

72. $b^2 + 14b + 48$

73. $a^2 + 25a + 100$

74. $u^2 + 101u + 100$

75. $x^2 - 8x + 12$

76. $q^2 - 13q + 36$

77. $y^2 - 18x + 45$

78. $m^2 - 13m + 30$

79. $x^2 - 8x + 7$

80. $y^2 - 5y + 6$

81. $p^2 + 5p - 6$

82. $n^2 + 6n - 7$

83. $y^2 - 6y - 7$

84. $v^2 - 2v - 3$

85. $x^2 - x - 12$

86. $r^2 - 2r - 8$

87. $a^2 - 3a - 28$

88. $b^2 - 13b - 30$

89. $w^2 - 5w - 36$

90. $t^2 - 3t - 54$

91. $x^2 + x + 5$

92. $x^2 - 3x - 9$

93. $8 - 6x + x^2$

94. $7x + x^2 + 6$

95. $x^2 - 12 - 11x$

96. $-11 - 10x + x^2$

Factor Trinomials of the Form $x^2 + bxy + cy^2$

In the following exercises, factor each trinomial of the form $x^2 + bxy + cy^2$.

97. $p^2 + 3pq + 2q^2$

98. $m^2 + 6mn + 5n^2$

99. $r^2 + 15rs + 36s^2$

100. $u^2 + 10uv + 24v^2$

101. $m^2 - 12mn + 20n^2$

102. $p^2 - 16pq + 63q^2$

103. $x^2 - 2xy - 80y^2$

104. $p^2 - 8pq - 65q^2$

105. $m^2 - 64mn - 65n^2$

106. $p^2 - 2pq - 35q^2$

107. $a^2 + 5ab - 24b^2$

108. $r^2 + 3rs - 28s^2$

109. $x^2 - 3xy - 14y^2$

110. $u^2 - 8uv - 24v^2$

111. $m^2 - 5mn + 30n^2$

112. $c^2 - 7cd + 18d^2$

Mixed Practice

In the following exercises, factor each expression.

113. $u^2 - 12u + 36$

114. $w^2 + 4w - 32$

115. $x^2 - 14x - 32$

116. $y^2 + 41y + 40$

117. $r^2 - 20rs + 64s^2$

118. $x^2 - 16xy + 64y^2$

119. $k^2 + 34k + 120$

120. $m^2 + 29m + 120$

121. $y^2 + 10y + 15$

122. $z^2 - 3z + 28$

123. $m^2 + mn - 56n^2$

124. $q^2 - 29qr - 96r^2$

125. $u^2 - 17uv + 30v^2$

126. $m^2 - 31mn + 30n^2$

127. $c^2 - 8cd + 26d^2$

128. $r^2 + 11rs + 36s^2$

Everyday Math

129. Consecutive integers Deirdre is thinking of two consecutive integers whose product is 56. The trinomial $x^2 + x - 56$ describes how these numbers are related. Factor the trinomial.

130. Consecutive integers Deshawn is thinking of two consecutive integers whose product is 182. The trinomial $x^2 + x - 182$ describes how these numbers are related. Factor the trinomial.

Writing Exercises

131. Many trinomials of the form $x^2 + bx + c$ factor into the product of two binomials $(x + m)(x + n)$. Explain how you find the values of *m* and *n*.

132. How do you determine whether to use plus or minus signs in the binomial factors of a trinomial of the form $x^2 + bx + c$ where b and c may be positive or negative numbers?

133. Will factored $x^2 - x - 20$ as $(x + 5)(x - 4)$. Bill factored it as $(x + 4)(x - 5)$. Phil factored it as $(x - 5)(x - 4)$. Who is correct? Explain why the other two are wrong.

134. Look at **Example 7.19**, where we factored $y^2 + 17y + 60$. We made a table listing all pairs of factors of 60 and their sums. Do you find this kind of table helpful? Why or why not?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
factor trinomials of the form $x^2 + bx + c$.			
factor trinomials of the form $x^2 + bxy + cy^2$.			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

7.3 Factor Quadratic Trinomials with Leading Coefficient Other than 1

Learning Objectives

By the end of this section, you will be able to:

- Recognize a preliminary strategy to factor polynomials completely
- Factor trinomials of the form $ax^2 + bx + c$ with a GCF
- Factor trinomials using trial and error
- Factor trinomials using the 'ac' method

Be Prepared!

Before you get started, take this readiness quiz.

1. Find the GCF of $45p^2$ and $30p^6$.
   If you missed this problem, review **Example 7.2**.
2. Multiply $(3y + 4)(2y + 5)$.
   If you missed this problem, review **Example 6.40**.
3. Combine like terms $12x^2 + 3x + 5x + 9$.
   If you missed this problem, review **Example 1.24**.

Recognize a Preliminary Strategy for Factoring

Let's summarize where we are so far with factoring polynomials. In the first two sections of this chapter, we used three methods of factoring: factoring the GCF, factoring by grouping, and factoring a trinomial by "undoing" FOIL. More methods will follow as you continue in this chapter, as well as later in your studies of algebra.

How will you know when to use each factoring method? As you learn more methods of factoring, how will you know when to apply each method and not get them confused? It will help to organize the factoring methods into a strategy that can guide you to use the correct method.

As you start to factor a polynomial, always ask first, "Is there a greatest common factor?" If there is, factor it first.

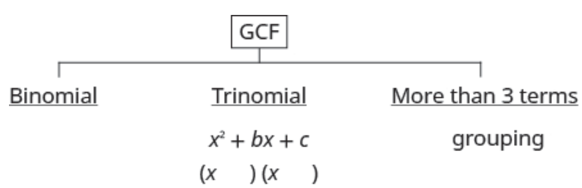
The next thing to consider is the type of polynomial. How many terms does it have? Is it a binomial? A trinomial? Or does it have more than three terms?

If it is a trinomial where the leading coefficient is one, $x^2 + bx + c$, use the "undo FOIL" method.

If it has more than three terms, try the grouping method. This is the only method to use for polynomials of more than three terms.

Some polynomials cannot be factored. They are called "prime."

Below we summarize the methods we have so far. These are detailed in **Choose a strategy to factor polynomials completely**.



HOW TO : : CHOOSE A STRATEGY TO FACTOR POLYNOMIALS COMPLETELY.

Step 1. Is there a greatest common factor?
- Factor it out.

Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?
- If it is a binomial, right now we have no method to factor it.
- If it is a trinomial of the form $x^2 + bx + c$: Undo FOIL $(x \quad)(x \quad)$
- If it has more than three terms: Use the grouping method.

Step 3. Check by multiplying the factors.

Use the preliminary strategy to completely factor a polynomial. A polynomial is factored completely if, other than monomials, all of its factors are prime.

EXAMPLE 7.29

Identify the best method to use to factor each polynomial.

ⓐ $6y^2 - 72$ ⓑ $r^2 - 10r - 24$ ⓒ $p^2 + 5p + pq + 5q$

Solution

ⓐ

	$6y^2 - 72$
Is there a greatest common factor?	Yes, 6.
Factor out the 6.	$6(y^2 - 12)$
Is it a binomial, trinomial, or are there more than 3 terms?	Binomial, we have no method to factor binomials yet.

ⓑ

	$r^2 - 10r - 24$
Is there a greatest common factor?	No, there is no common factor.
Is it a binomial, trinomial, or are there more than three terms?	Trinomial, with leading coefficient 1, “undo” FOIL.

ⓒ

	$p^2 + 5p + pq + 5q$
Is there a greatest common factor?	No, there is no common factor.
Is it a binomial, trinomial, or are there more than three terms?	More than three terms, so factor using grouping.

TRY IT : : 7.57 Identify the best method to use to factor each polynomial:

ⓐ $4y^2 + 32$ ⓑ $y^2 + 10y + 21$ ⓒ $yz + 2y + 3z + 6$

TRY IT : : 7.58 Identify the best method to use to factor each polynomial:

ⓐ $ab + a + 4b + 4$ ⓑ $3k^2 + 15$ ⓒ $p^2 + 9p + 8$

Factor Trinomials of the form $ax^2 + bx + c$ with a GCF

Now that we have organized what we've covered so far, we are ready to factor trinomials whose leading coefficient is not 1, trinomials of the form $ax^2 + bx + c$.

Remember to always check for a GCF first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods in the last section. Let's do a few examples to see how this works.

Watch out for the signs in the next two examples.

EXAMPLE 7.30

Factor completely: $2n^2 - 8n - 42$.

Solution

Use the preliminary strategy.

Is there a greatest common factor?	$2n^2 - 8n - 42$
Yes, GCF = 2. Factor it out.	$2\left(n^2 - 4n - 21\right)$
Inside the parentheses, is it a binomial, trinomial, or are there more than three terms?	
It is a trinomial whose coefficient is 1, so undo OIL.	$2(n \quad)(n \quad)$
Use 3 and −7 as the last terms of the binomials.	$2(n + 3)(n - 7)$

Factors of -21	Sum of factors
$1, -21$	$1 + (-21) = -20$
$3, -7$	$3 + (-7) = -4^*$

Check.

$$2(n + 3)(n - 7)$$

$$2(n^2 - 7n + 3n - 21)$$

$$2(n^2 - 4n - 21)$$

$$2n^2 - 8n - 42 \checkmark$$

TRY IT : : 7.59 Factor completely: $4m^2 - 4m - 8$.

TRY IT : : 7.60 Factor completely: $5k^2 - 15k - 50$.

EXAMPLE 7.31

Factor completely: $4y^2 - 36y + 56$.

Solution

Use the preliminary strategy.

Is there a greatest common factor?	$4y^2 - 36y + 56$
Yes, GCF = 4. Factor it.	$4\left(y^2 - 9y + 14\right)$
Inside the parentheses, is it a binomial, trinomial, or are there more than three terms? It is a trinomial whose coefficient is 1. So undo OIL.	$4(y \quad)(y \quad)$
Use a table like the one below to find t o numbers that multiply to 14 and add to -9. Both factors of 14 must be negative.	$4(y-2)(y-7)$

Factors of 14	Sum of factors
$-1, -14$	$-1 + (-14) = -15$
$-2, -7$	$-2 + (-7) = -9$*

Check.

$4(y-2)(y-7)$

$4(y^2 - 7y - 2y + 14)$

$4(y^2 - 9y + 14)$

$4y^2 - 36y + 42$ ✓

TRY IT : : 7.61 Factor completely: $3r^2 - 9r + 6$.

TRY IT : : 7.62 Factor completely: $2t^2 - 10t + 12$.

In the next example the GCF will include a variable.

EXAMPLE 7.32

Factor completely: $4u^3 + 16u^2 - 20u$.

Solution

Use the preliminary strategy.

Is there a greatest common factor?	$4u^3 + 16u^2 - 20u$
Yes, GCF $= 4u$. Factor it.	$4u\left(u^2 + 4u - 5\right)$
Binomial, trinomial, or more than three terms? It is a trinomial. So "undo FOIL."	$4u(u \quad)(u \quad)$
Use a table like the table below to find t o numbers that multiply to -5 and add to 4.	$4u(u - 1)(u + 5)$

Factors of -5	Sum of factors
$-1, 5$	$-1 + 5 = 4^*$
$1, -5$	$1 + (-5) = -4$

Check.

$$4u(u - 1)(u + 5)$$
$$4u(u^2 + 5u - u - 5)$$
$$4u(u^2 + 4u - 5)$$
$$4u^3 + 16u^2 - 20u \checkmark$$

TRY IT : : 7.63 Factor completely: $5x^3 + 15x^2 - 20x$.

TRY IT : : 7.64 Factor completely: $6y^3 + 18y^2 - 60y$.

Factor Trinomials using Trial and Error

What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial $3x^2 + 5x + 2$.

From our earlier work we expect this will factor into two binomials.

$$3x^2 + 5x + 2$$
$$(\quad)(\quad)$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x, 3x$. We can place them in the binomials.

$$3x^2 + 5x + 2$$
$$1x,\ 3x$$
$$(x \quad)(3x \quad)$$

Check. Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1 and 2. But we now have two cases to consider as it will make a difference if we write 1, 2, or 2, 1.

$$\begin{array}{ccc} 3x^2 + 5x + 2 & & 3x^2 + 5x + 2 \\ 1x,\ 3x \quad\quad 1,\ 2 & & 1x,\ 3x \quad\quad 1,\ 2 \\ (x+1)(3x+2) & \text{or} & (x+2)(3x+1) \end{array}$$

Which factors are correct? To decide that, we multiply the inner and outer terms.

$$\begin{array}{ccc} 3x^2 + 5x + 2 & & 3x^2 + 5x + 2 \\ 1x,\ 3x \quad\quad 1,\ 2 & & 1x,\ 3x \quad\quad 1,\ 2 \\ (x+1)(3x+2) & \text{or} & (x+2)(3x+1) \\ 3x & & 6x \\ \underline{2x} & & \underline{1x} \\ 5x & & 7x \end{array}$$

Since the middle term of the trinomial is $5x$, the factors in the first case will work. Let's FOIL to check.

$$\begin{array}{l} (x+1)(3x+2) \\ 3x^2 + 2x + 3x + 2 \\ 3x^2 + 5x + 2 \ \checkmark \end{array}$$

Our result of the factoring is:

$$\begin{array}{c} 3x^2 + 5x + 2 \\ (x+1)(3x+2) \end{array}$$

EXAMPLE 7.33 HOW TO FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING TRIAL AND ERROR

Factor completely: $3y^2 + 22y + 7$.

Solution

<table>
<tr><td>Step 1. Write the trinomial in descending order.</td><td>The trinomial is already in descending order.</td><td>$3y^2 + 22y + 7$</td></tr>
<tr><td>Step 2. Find all the factor pairs of the first term.</td><td>The only factors of $3y^2$ are $1y$, $3y$

Since there is only one pair, we can put them in the parentheses.</td><td>$3y^2 + 22y + 7$
$1y,\ 3y$

$3y^2 + 22y + 7$
$1y,\ 3y$
$(y\quad)(3y\quad)$</td></tr>
<tr><td>Step 3. Find all the factor pairs of the third term.</td><td>The only factors of 7 are 1, 7.</td><td>$3y^2 + 22y + 7$
$1y,\ 3y \quad 1,\ 7$
$(y\quad)(3y\quad)$</td></tr>
<tr><td>Step 4. Test all the possible combinations of the factors until the correct product is found.</td><td>$3y^2 + 22y + 7$
$1y,\ 3y \quad 1,\ 7$
$(y+1)(3y+7)$
$3y$
$\underline{7y}$
$10y$
No. We need $22y$

$3y^2 + 22y + 7$
$1y,\ 3y \quad 1,\ 7$
$(y+7)(3y+1)$
$21y$
$\underline{y}$
$22y$</td><td><table><tr><th colspan="2">$3y^2 + 22y + 7$</th></tr><tr><th>Possible factors</th><th>Product</th></tr><tr><td>$(y+1)(3y+7)$</td><td>$3y^2 + 10y + 7$</td></tr><tr><td>$(y+7)(3y+1)$</td><td>$3y^2 + 22y + 7$</td></tr></table>
$(y+7)(3y+1)$</td></tr>
</table>

Step 5. Check by multiplying.		$(y+7)(3y+1)$ $3y^2+22y+7$ ✓

TRY IT : : 7.65 Factor completely: $2a^2+5a+3$.

TRY IT : : 7.66 Factor completely: $4b^2+5b+1$.

HOW TO : : FACTOR TRINOMIALS OF THE FORM ax^2+bx+c USING TRIAL AND ERROR.

Step 1. Write the trinomial in descending order of degrees.
Step 2. Find all the factor pairs of the first term.
Step 3. Find all the factor pairs of the third term.
Step 4. Test all the possible combinations of the factors until the correct product is found.
Step 5. Check by multiplying.

When the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

EXAMPLE 7.34

Factor completely: $6b^2-13b+5$.

Solution

The trinomial is already in descending order.	$6b^2-13b+5$
Find the factors of the first term.	$6b^2-13b+5$ $1b\cdot 6b$ $2b\cdot 3b$
Find the factors of the last term. Consider the signs. Since the last term, 5 is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	$6b^2-13b+5$ $1b\cdot 6b$ $-1,-5$ $2b\cdot 3b$

Consider all the combinations of factors.

$6b^2-13b+5$	
Possible factors	**Product**
$(b-1)(6b-5)$	$6b^2-11b+5$
$(b-5)(6b-1)$	$6b^2-31b+5$
$(2b-1)(3b-5)$	$6b^2-13b+5$ *
$(2b-5)(3b-1)$	$6b^2-17b+5$

The correct factors are those whose product is the original trinomial.	$(2b-1)(3b-5)$

Check by multiplying.

$$(2b-1)(3b-5)$$
$$6b^2-10b-3b+5$$
$$6b^2-13b+5\ \checkmark$$

TRY IT : : 7.67 Factor completely: $8x^2-13x+3$.

TRY IT : : 7.68 Factor completely: $10y^2-37y+7$.

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

EXAMPLE 7.35

Factor completely: $14x^2-47x-7$.

Solution

The trinomial is already in descending order.	$14x^2-47x-7$
Find the factors of the first term.	$14x^2-47x-7$ $1x \cdot 14x$ $2x \cdot 7x$
Find the factors of the last term. Consider the signs. Since it is negative, one factor must be positive and one negative.	$14x^2-47x-7$ $1x \cdot 14x$ $1, -7$ $2x \cdot 7x$ $-1, 7$

Consider all the combinations of factors. We use each pair of the factors of $14x^2$ with each pair of factors of -7.

Factors of $14x^2$	Pair with	Factors of -7
x , $14x$		1 , -7 -7 , 1 (reverse order)
x , $14x$		-1 , 7 7 , -1 (reverse order)
$2x$, $7x$		1 , -7 -7 , 1 (reverse order)
$2x$, $7x$		-1 , 7 7 , -1 (reverse order)

These pairings lead to the following eight combinations.

$14x^2 - 47x - 7$	
Possible factors	**Product**
$(x + 1)(14x - 7)$	Not an option
$(x - 7)(14x + 1)$	$14x^2 - 97x - 7$
$(x - 1)(14x + 7)$	Not an option
$(x + 7)(14x - 1)$	$14x^2 + 97x - 7$
$(2x + 1)(7x - 7)$	Not an option
$(2x - 7)(7x + 1)$	$14x^2 - 47x - 7$*
$(2x - 1)(7x + 7)$	Not an option
$(2x + 7)(7x - 1)$	$14x^2 + 47x - 7$

If the trinomial has no common factors, then neither factor can contain a common factor. That means each of these combinations is not an option.

The correct factors are those whose product is the original trinomial.	$(2x - 7)(7x + 1)$
Check by multiplying.	

$$(2x - 7)(7x + 1)$$
$$14x^2 + 2x - 49x - 7$$
$$14x^2 - 47x - 7 \checkmark$$

TRY IT : : 7.69 Factor completely: $8a^2 - 3a - 5$.

TRY IT : : 7.70 Factor completely: $6b^2 - b - 15$.

EXAMPLE 7.36

Factor completely: $18n^2 - 37n + 15$.

Solution

The trinomial is already in descending order.	$18n^2 - 37n + 15$
Find the factors of the first term.	$18n^2 - 37n + 15$ $1n \cdot 18n$ $2n \cdot 9n$ $3n \cdot 6n$
Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative facotrs.	$18n^2 - 37n + 15$ $1n \cdot 18n$ $\quad -1(-15)$ $2n \cdot 9n$ $\quad -3(-5)$ $3n \cdot 6n$

Consider all the combinations of factors.

$18n^2 - 37n + 15$	
Possible factors	**Product**
$(n - 1)(18n - 15)$	Not an option
$(n - 15)(18n - 1)$	$18n^2 - 271n + 15$
$(n - 3)(18n - 5)$	$18n^2 - 59n + 15$
$(n - 5)(18n - 3)$	Not an option
$(2n - 1)(9n - 15)$	Not an option
$(2n - 15)(9n - 1)$	$18n^2 - 137n + 15$
$(2n - 3)(9n - 5)$	$18n^2 - 37n + 15$*
$(2n - 5)(9n - 3)$	Not an option
$(3n - 1)(6n - 15)$	Not an option
$(3n - 15)(6n - 1)$	Not an option
$(3n - 3)(6n - 5)$	Not an option
$(3n - 5)(6n - 3)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. $(2n-3)(9n-5)$

Check by multiplying.

$$(2n-3)(9n-5)$$
$$18n^2-10n-27n+15$$
$$18n^2-37n+15\ \checkmark$$

TRY IT : : 7.71 Factor completely: $18x^2-3x-10$.

TRY IT : : 7.72 Factor completely: $30y^2-53y-21$.

Don't forget to look for a GCF first.

EXAMPLE 7.37

Factor completely: $10y^4+55y^3+60y^2$.

Solution

	$10y^4+55y^3+60y^2$
Notice the greatest common factor, and factor it first.	$15y^2\left(2y^2+11y+12\right)$
Factor the trinomial.	$5y^2(\underbrace{2y^2}_{y\cdot 2y}+11y+\underbrace{12}_{\substack{1\cdot 12\\ 2\cdot 6\\ 3\cdot 4}})$

Consider all the combinations.

$2y^2+11y+12$	
Possible factors	**Product**
$(y+1)(2y+12)$	Not an option
$(y+12)(2y+1)$	$2y^2+25y+12$
$(y+2)(2y+6)$	Not an option
$(y+6)(2y+2)$	Not an option
$(y+3)(2y+4)$	Not an option
$(y+4)(2y+3)$	$2y^2+11y+12$*

> If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. Remember to include the factor $5y^2$. $5y^2(y+4)(2y+3)$

Check by multiplying.

$$5y^2(y+4)(2y+3)$$
$$5y^2\left(2y^2+8y+3y+12\right)$$
$$10y^4+55y^3+60y^2\ \checkmark$$

TRY IT : : 7.73 Factor completely: $15n^3-85n^2+100n$.

TRY IT : : 7.74 Factor completely: $56q^3 + 320q^2 - 96q$.

Factor Trinomials using the "ac" Method

Another way to factor trinomials of the form $ax^2 + bx + c$ is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

EXAMPLE 7.38 HOW TO FACTOR TRINOMIALS USING THE "AC" METHOD

Factor: $6x^2 + 7x + 2$.

Solution

Step 1. Factor any GCF.	Is there a greatest common factor? No.	$6x^2 + 7x + 2$
Step 2. Find the product *ac*.	$a \cdot c$ $6 \cdot 2$ 12	$ax^2 + bx + c$ $6x^2 + 7x + 2$
Step 3. Find two numbers *m* and *n* that: Multiply to *ac* $m \cdot n = a \cdot c$ Add to *b* $m + n = b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12$ $3 + 4 = 7$	
Step 4. Split the middle term using *m*, and *n* $ax^2 + bx + c$ $\underbrace{ax^2 + mx + nx}_{}$ bx $+ c$	Rewrite $7x$ as $3x + 4x$. Notice that $6x^2 + 3x + 4x + 2$ is equal to $6x^2 + 7x + 2$. We just split the middle term to get a more useful form.	$6x^2 + 7x + 2$ $\underbrace{6x^2 + 3x}_{} + \underbrace{4x + 2}_{}$
Step 5. Factor by grouping.		$3x(2x + 1) + 2(2x + 1)$ $(2x + 1)(3x + 2)$
Step 6. Check by multiplying.		$(2x + 1)(3x + 2)$ $6x^2 + 4x + 3x + 2$ $6x^2 + 7x + 2$ ✓

TRY IT : : 7.75 Factor: $6x^2 + 13x + 2$.

TRY IT : : 7.76 Factor: $4y^2 + 8y + 3$.

HOW TO : : FACTOR TRINOMIALS OF THE FORM USING THE "AC" METHOD.

Step 1. Factor any GCF.

Step 2. Find the product ac.

Step 3. Find two numbers m and n that:

Multiply to ac	$m \cdot n = a \cdot c$
Add to b	$m + n = b$

Step 4. Split the middle term using m and n:

$$\begin{array}{c} ax^2 + \underbrace{bx}_{} + c \\ ax^2 + \overbrace{mx + nx}^{bx} + c \end{array}$$

Step 5. Factor by grouping.

Step 6. Check by multiplying the factors.

When the third term of the trinomial is negative, the factors of the third term will have opposite signs.

EXAMPLE 7.39

Factor: $8u^2 - 17u - 21$.

Solution

Is there a greatest common factor? No.	$\overset{ax^2}{8u^2} \overset{+\ bx}{-\ 17u} \overset{+\ c}{-\ 21}$
Find $a \cdot c$.	$a \cdot c$
	$8(-21)$
	-168

Find two numbers that multiply to -168 and add to -17 . The larger factor must be negative.

Factors of -168	Sum of factors
$1, -168$	$1 + (-168) = -167$
$2, -84$	$2 + (-84) = -82$
$3, -56$	$3 + (-56) = -53$
$4, -42$	$4 + (-42) = -38$
$6, -28$	$6 + (-28) = -22$
$7, -24$	$7 + (-24) = -17$*
$8, -21$	$8 + (-21) = -13$

Split the middle term using $7u$ and $-24u$.

$$8u^2 - 17u - 21$$

$$\underbrace{8u^2 + 7u}\underbrace{-24u - 21}$$

Factor by grouping.

$$u(8u + 7) - 3(8u + 7)$$
$$(8u + 7)(u - 3)$$

Check by multiplying.

$$(8u + 7)(u - 3)$$
$$8u^2 - 24u + 7u - 21$$
$$8u^2 - 17u - 21 \checkmark$$

> **TRY IT : : 7.77** Factor: $20h^2 + 13h - 15$.

> **TRY IT : : 7.78** Factor: $6g^2 + 19g - 20$.

EXAMPLE 7.40

Factor: $2x^2 + 6x + 5$.

Solution

Is there a greatest common factor? No.	$\overset{ax^2 + bx + c}{2x^2 + 6x + 5}$
Find $a \cdot c$.	$a \cdot c$
	$2(5)$
	10

Find two numbers that multiply to 10 and add to 6.

Factors of 10	Sum of factors
$1, 10$	$1 + 10 = 11$
2, 5	$2 + 5 = 7$

There are no factors that multiply to 10 and add to 6. The polynomial is prime.

> **TRY IT : : 7.79** Factor: $10t^2 + 19t - 15$.

> **TRY IT : : 7.80** Factor: $3u^2 + 8u + 5$.

Don't forget to look for a common factor!

EXAMPLE 7.41

Factor: $10y^2 - 55y + 70$.

Solution

Is there a greatest common factor? Yes. The GCF is 5.	$10y^2 - 55y + 70$
Factor it. Be careful to keep the factor of 5 all the way through the solution!	$5(2y^2 - 11y + 14)$
The trinomial inside the parentheses has a leading coefficient that is not 1.	$ax^2 + bx + c$ $5(2y^2 - 11y + 14)$
Factor the trinomial.	$5(y - 2)(2y - 7)$
Check by mulitplying all three factors.	
$5\left(2y^2 - 2y - 4y + 14\right)$	
$5\left(2y^2 - 11y + 14\right)$	
$10y^2 - 55y + 70$ ✓	

TRY IT : : 7.81 Factor: $16x^2 - 32x + 12$.

TRY IT : : 7.82 Factor: $18w^2 - 39w + 18$.

We can now update the Preliminary Factoring Strategy, as shown in **Figure 7.2** and detailed in **Choose a strategy to factor polynomials completely (updated)**, to include trinomials of the form $ax^2 + bx + c$. Remember, some polynomials are prime and so they cannot be factored.

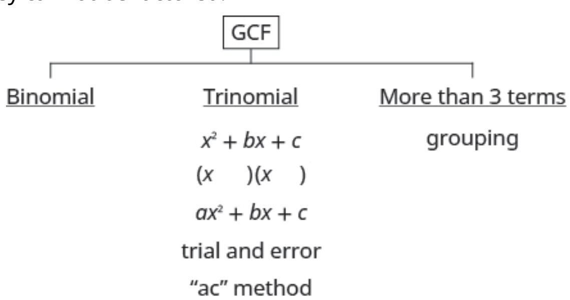


Figure 7.2

HOW TO : : CHOOSE A STRATEGY TO FACTOR POLYNOMIALS COMPLETELY (UPDATED).

Step 1. Is there a greatest common factor?
- Factor it.

Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?
- If it is a binomial, right now we have no method to factor it.
- If it is a trinomial of the form $x^2 + bx + c$
  Undo FOIL $(x \quad)(x \quad)$.
- If it is a trinomial of the form $ax^2 + bx + c$
  Use Trial and Error or the "ac" method.
- If it has more than three terms
  Use the grouping method.

Step 3. Check by multiplying the factors.

MEDIA : :

Access these online resources for additional instruction and practice with factoring trinomials of the form $ax^2 + bx + c$.

- **Factoring Trinomials, a is not 1 (https://openstax.org/l/25FactorTrinom)**

7.3 EXERCISES

Practice Makes Perfect

Recognize a Preliminary Strategy to Factor Polynomials Completely

In the following exercises, identify the best method to use to factor each polynomial.

135.
ⓐ $10q^2 + 50$
ⓑ $a^2 - 5a - 14$
ⓒ $uv + 2u + 3v + 6$

136.
ⓐ $n^2 + 10n + 24$
ⓑ $8u^2 + 16$
ⓒ $pq + 5p + 2q + 10$

137.
ⓐ $x^2 + 4x - 21$
ⓑ $ab + 10b + 4a + 40$
ⓒ $6c^2 + 24$

138.
ⓐ $20x^2 + 100$
ⓑ $uv + 6u + 4v + 24$
ⓒ $y^2 - 8y + 15$

Factor Trinomials of the form $ax^2 + bx + c$ with a GCF

In the following exercises, factor completely.

139. $5x^2 + 35x + 30$

140. $12s^2 + 24s + 12$

141. $2z^2 - 2z - 24$

142. $3u^2 - 12u - 36$

143. $7v^2 - 63v + 56$

144. $5w^2 - 30w + 45$

145. $p^3 - 8p^2 - 20p$

146. $q^3 - 5q^2 - 24q$

147. $3m^3 - 21m^2 + 30m$

148. $11n^3 - 55n^2 + 44n$

149. $5x^4 + 10x^3 - 75x^2$

150. $6y^4 + 12y^3 - 48y^2$

Factor Trinomials Using Trial and Error

In the following exercises, factor.

151. $2t^2 + 7t + 5$

152. $5y^2 + 16y + 11$

153. $11x^2 + 34x + 3$

154. $7b^2 + 50b + 7$

155. $4w^2 - 5w + 1$

156. $5x^2 - 17x + 6$

157. $6p^2 - 19p + 10$

158. $21m^2 - 29m + 10$

159. $4q^2 - 7q - 2$

160. $10y^2 - 53y - 11$

161. $4p^2 + 17p - 15$

162. $6u^2 + 5u - 14$

163. $16x^2 - 32x + 16$

164. $81a^2 + 153a - 18$

165. $30q^3 + 140q^2 + 80q$

166. $5y^3 + 30y^2 - 35y$

Factor Trinomials using the 'ac' Method

In the following exercises, factor.

167. $5n^2 + 21n + 4$

168. $8w^2 + 25w + 3$

169. $9z^2 + 15z + 4$

170. $3m^2 + 26m + 48$

171. $4k^2 - 16k + 15$

172. $4q^2 - 9q + 5$

173. $5s^2 - 9s + 4$

174. $4r^2 - 20r + 25$

175. $6y^2 + y - 15$

176. $6p^2 + p - 22$

177. $2n^2 - 27n - 45$

178. $12z^2 - 41z - 11$

179. $3x^2 + 5x + 4$

180. $4y^2 + 15y + 6$

181. $60y^2 + 290y - 50$

182. $6u^2 - 46u - 16$

183. $48z^3 - 102z^2 - 45z$

184. $90n^3 + 42n^2 - 216n$

185. $16s^2 + 40s + 24$

186. $24p^2 + 160p + 96$

187. $48y^2 + 12y - 36$

188. $30x^2 + 105x - 60$

Mixed Practice

In the following exercises, factor.

189. $12y^2 - 29y + 14$

190. $12x^2 + 36y - 24z$

191. $a^2 - a - 20$

192. $m^2 - m - 12$

193. $6n^2 + 5n - 4$

194. $12y^2 - 37y + 21$

195. $2p^2 + 4p + 3$

196. $3q^2 + 6q + 2$

197. $13z^2 + 39z - 26$

198. $5r^2 + 25r + 30$

199. $x^2 + 3x - 28$

200. $6u^2 + 7u - 5$

201. $3p^2 + 21p$

202. $7x^2 - 21x$

203. $6r^2 + 30r + 36$

204. $18m^2 + 15m + 3$

205. $24n^2 + 20n + 4$

206. $4a^2 + 5a + 2$

207. $x^2 + 2x - 24$

208. $2b^2 - 7b + 4$

Everyday Math

209. Height of a toy rocket The height of a toy rocket launched with an initial speed of 80 feet per second from the balcony of an apartment building is related to the number of seconds, *t*, since it is launched by the trinomial $-16t^2 + 80t + 96$. Completely factor the trinomial.

210. Height of a beach ball The height of a beach ball tossed up with an initial speed of 12 feet per second from a height of 4 feet is related to the number of seconds, *t*, since it is tossed by the trinomial $-16t^2 + 12t + 4$. Completely factor the trinomial.

Writing Exercises

211. List, in order, all the steps you take when using the "ac" method to factor a trinomial of the form $ax^2 + bx + c$.

212. How is the "ac" method similar to the "undo FOIL" method? How is it different?

213. What are the questions, in order, that you ask yourself as you start to factor a polynomial? What do you need to do as a result of the answer to each question?

214. On your paper draw the chart that summarizes the factoring strategy. Try to do it without looking at the book. When you are done, look back at the book to finish it or verify it.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
recognize a preliminary strategy to factor polynomials completely.			
factor trinomials of the form $ax^2 + bx + c$ with a GCF.			
factor trinomials using trial and error.			
factor trinomials using the "ac" method.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

7.4 Factor Special Products

Learning Objectives

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes
- Choose method to factor a polynomial completely

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $(12x)^2$.
   If you missed this problem, review Example 6.23.
2. Multiply: $(m+4)^2$.
   If you missed this problem, review Example 6.47.
3. Multiply: $(p-9)^2$.
   If you missed this problem, review Example 6.48.
4. Multiply: $(k+3)(k-3)$.
   If you missed this problem, review Example 6.52.

The strategy for factoring we developed in the last section will guide you as you factor most binomials, trinomials, and polynomials with more than three terms. We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. You can square a binomial by using FOIL, but using the Binomial Squares pattern you saw in a previous chapter saves you a step. Let's review the Binomial Squares pattern by squaring a binomial using FOIL.

$$(3x+4)^2$$
$$(3x+4)(3x+4)$$
$$\overset{F}{9x^2} + \overset{O}{12x} + \overset{I}{12x} + \overset{L}{16}$$
$$9x^2+24x+16$$

The first term is the square of the first term of the binomial and the last term is the square of the last. The middle term is twice the product of the two terms of the binomial.

$$\begin{gathered}(3x)^2+2(3x\cdot 4)+4^2\\ 9x^2+24x+16\end{gathered}$$

The trinomial $9x^2 + 24 + 16$ is called a perfect square trinomial. It is the square of the binomial $3x+4$.

We'll repeat the Binomial Squares Pattern here to use as a reference in factoring.

Binomial Squares Pattern

If a and b are real numbers,

$$(a+b)^2=a^2+2ab+b^2 \qquad (a-b)^2=a^2-2ab+b^2$$

When you square a binomial, the product is a perfect square trinomial. In this chapter, you are learning to factor—now, you will start with a perfect square trinomial and factor it into its prime factors.

You could factor this trinomial using the methods described in the last section, since it is of the form $ax^2 + bx + c$. But if you recognize that the first and last terms are squares and the trinomial fits the **perfect square trinomials pattern**, you will save yourself a lot of work.

Here is the pattern—the reverse of the binomial squares pattern.

Perfect Square Trinomials Pattern

If a and b are real numbers,

$$a^2 + 2ab + b^2 = (a + b)^2 \qquad a^2 - 2ab + b^2 = (a - b)^2$$

To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it twice the product, $2ab$? If everything checks, you can easily write the factors.

EXAMPLE 7.42 HOW TO FACTOR PERFECT SQUARE TRINOMIALS

Factor: $9x^2 + 12x + 4$.

Solution

Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$?		
• Is the first term a perfect square? Write it as a square, a^2.	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$.	$9x^2 + 12x + 4$ $(3x)^2$
• Is the last term a perfect square? Write it as a square, b^2.	Is 4 a perfect square? Yes—write it as $(2)^2$.	$(3x)^2 \qquad (2)^2$
• Check the middle term. Is it $2ab$?	Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial!	$(3x)^2 \qquad (2)^2$ $2(3x)(2)$ $12x$
Step 2. Write the square of the binomial.	Write it as the square of a binomial.	$9x^2 + 12x + 4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $(a + b)^2$ $(3x + 2)^2$
Step 3. Check. $(3x + 2)^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $9x^2 + 12x + 4$ ✓		

> **TRY IT : :** 7.83 Factor: $4x^2 + 12x + 9$.

> **TRY IT : :** 7.84 Factor: $9y^2 + 24y + 16$.

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2 - 2ab + b^2$, which factors to $(a - b)^2$.

The steps are summarized here.

HOW TO : : FACTOR PERFECT SQUARE TRINOMIALS.

Step 1. Does the trinomial fit he pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
• Is the fir t term a perfect square? Write it as a square.	$(a)^2$	$(a)^2$
• Is the last term a perfect square? Write it as a square.	$(a)^2 \quad (b)^2$	$(a)^2 \quad (b)^2$
• Check the middle term. Is it $2ab$?	$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$	$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$
Step 2. Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3. Check by multiplying.		

We'll work one now where the middle term is negative.

EXAMPLE 7.43

Factor: $81y^2 - 72y + 16$.

Solution

The first and last terms are squares. See if the middle term fits the pattern of a perfect square trinomial. The middle term is negative, so the binomial square would be $(a-b)^2$.

	$81y^2 - 72y + 16$
Are the first and last terms perfect squares?	$(9y)^2 \quad (4)^2$
Check the middle term.	$(9y)^2 \quad (4)^2$ $2(9y)(4)$ $72y$
Does is match $(a-b)^2$? Yes.	$\overset{a^2}{(9y)^2} \overset{-2}{-2} \cdot \overset{a}{9y} \cdot \overset{b}{4} \overset{+}{+} \overset{b^2}{4^2}$
Write the square of a binomial.	$(9y-4)^2$
Check by mulitplying.	
$(9y-4)^2$	
$(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$	
$81y^2 - 72y + 16$ ✓	

TRY IT : : 7.85 Factor: $64y^2 - 80y + 25$.

TRY IT : : 7.86 Factor: $16z^2 - 72z + 81$.

The next example will be a perfect square trinomial with two variables.

EXAMPLE 7.44

Factor: $36x^2 + 84xy + 49y^2$.

Solution

	$36x^2 + 84xy + 49y^2$
Test each term to verify the pattern.	$\overset{a^2}{(6x)^2} \overset{+\,2}{+\,2} \cdot \overset{a}{6x} \cdot \overset{b}{7y} \overset{+}{+} \overset{b^2}{(7y)^2}$
Factor.	$(6x + 7y)^2$
Check by mulitplying.	
$(6x + 7y)^2$	
$(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$	
$36x^2 + 84xy + 49y^2$ ✓	

TRY IT : : 7.87 Factor: $49x^2 + 84xy + 36y^2$.

TRY IT : : 7.88 Factor: $64m^2 + 112mn + 49n^2$.

EXAMPLE 7.45

Factor: $9x^2 + 50x + 25$.

Solution

	$9x^2 + 50x + 25$
Are the fir t and last terms perfect squares?	$(3x)^2 \qquad (5)^2$
Check the middle term—is it $2ab$?	$(3x)^2 \searrow \underset{30x}{2(3x)(5)} \swarrow (5)^2$
No! $30x \neq 50x$	This does not fit he pattern!
Factor using the “ac” method.	$9x^2 + 50x + 25$

Notice: $\overset{ac}{\underset{225}{9 \cdot 25}}$ and $\begin{array}{l} 5 \cdot 45 = 225 \\ 5 + 45 = 50 \end{array}$

Split the middle term.	$9x^2 + 5x + 45x + 25$
Factor by grouping.	$x(9x + 5) + 5(9x + 5)$
	$(9x + 5)(x + 5)$

Check.

$(9x + 5)(x + 5)$

$9x^2 + 45x + 5x + 25$

$9x^2 + 50x + 25$ ✓

TRY IT : : 7.89 Factor: $16r^2 + 30rs + 9s^2$.

TRY IT : : 7.90 Factor: $9u^2 + 87u + 100$.

Remember the very first step in our Strategy for Factoring Polynomials? It was to ask “is there a greatest common factor?” and, if there was, you factor the GCF before going any further. Perfect square trinomials may have a GCF in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

EXAMPLE 7.46

Factor: $36x^2y - 48xy + 16y$.

Solution

	$36x^2y - 48xy + 16y$
Is there a GCF? Yes, 4*y*, so factor it out.	$4y\left(9x^2 - 12x + 4\right)$
Is this a perfect square trinomial?	
Verify the pattern.	$4y[\overset{a^2}{(3x)^2} \overset{-\;2}{-2} \cdot \overset{a}{3x} \cdot \overset{b}{2} \overset{+}{+} \overset{b^2}{2^2}]$
Factor.	$4y(3x - 2)^2$
Remember: Keep the factor 4*y* in the final product.	
Check.	
$4y(3x - 2)^2$	
$4y\left[(3x)^2 - 2 \cdot 3x \cdot 2 + 2^2\right]$	
$4y(9x)^2 - 12x + 4$	
$36x^2y - 48xy + 16y$ ✓	

TRY IT : : 7.91 Factor: $8x^2y - 24xy + 18y$.

TRY IT : : 7.92 Factor: $27p^2q + 90pq + 75q$.

Factor Differences of Squares

The other special product you saw in the previous was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here’s an example:

$$(3x - 4)(3x + 4)$$
$$9x^2 - 16$$

Remember, when you multiply conjugate binomials, the middle terms of the product add to 0. All you have left is a binomial, the difference of squares.

Multiplying conjugates is the only way to get a binomial from the product of two binomials.

Product of Conjugates Pattern

If *a* and *b* are real numbers

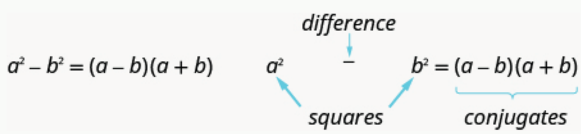
$$(a - b)(a + b) = a^2 - b^2$$

The product is called a difference of squares.

To factor, we will use the product pattern “in reverse” to factor the difference of squares. A **difference of squares** factors to a product of conjugates.

Difference of Squares Pattern

If a and b are real numbers,



Remember, “difference” refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

EXAMPLE 7.47 HOW TO FACTOR DIFFERENCES OF SQUARES

Factor: $x^2 - 4$.

Solution

Step 1. Does the binomial fit the pattern? • Is this a difference? • Are the first and last terms perfect squares?	 Yes Yes	$x^2 - 4$ $x^2 - 4$
Step 2. Write them as squares.	Write them as x^2 and 2^2.	$a^2 - b^2$ $(x)^2 - 2^2$
Step 3. Write the product of conjugates.		$(a - b)(a + b)$ $(x - 2)(x + 2)$
Step 4. Check.		$(x - 2)(x + 2)$ $x^2 - 4$ ✓

TRY IT : : 7.93 Factor: $h^2 - 81$.

TRY IT : : 7.94 Factor: $k^2 - 121$.

HOW TO : : FACTOR DIFFERENCES OF SQUARES.

Step 1. Does the binomial fit he pattern? $a^2 - b^2$
- Is this a diffe ence? $\_\_\_\_ - \_\_\_\_$
- Are the fir t and last terms perfect squares?

Step 2. Write them as squares. $(a)^2 - (b)^2$

Step 3. Write the product of conjugates. $(a - b)(a + b)$

Step 4. Check by multiplying.

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression $a^2 + b^2$ is prime!

Don't forget that 1 is a perfect square. We'll need to use that fact in the next example.

EXAMPLE 7.48

Factor: $64y^2 - 1$.

Solution

	$64y^2 - 1$
Is this a difference? Yes.	$64y^2 - 1$
Are the first and last terms perfect squares?	
Yes - write them as squares.	$\overset{a^2}{(8y)^2} - \overset{b^2}{1^2}$
Factor as the product of conjugates.	$\overset{(a-b)}{(8y-1)}\overset{(a+b)}{(8y+1)}$
Check by multiplying.	
$(8y - 1)(8y + 1)$	
$64y^2 - 1$ ✓	

TRY IT : : 7.95 Factor: $m^2 - 1$.

TRY IT : : 7.96 Factor: $81y^2 - 1$.

EXAMPLE 7.49

Factor: $121x^2 - 49y^2$.

Solution

	$121x^2 - 49y^2$
Is this a diffe ence of squares? Yes.	$(11x)^2 - (7y)^2$
Factor as the product of conjugates.	$(11x - 7y)(11x + 7y)$

Check by multiplying.

$(11x - 7y)(11x + 7y)$

$121x^2 - 49y^2$ ✓

TRY IT : : 7.97 Factor: $196m^2 - 25n^2$.

TRY IT : : 7.98 Factor: $144p^2 - 9q^2$.

The binomial in the next example may look “backwards,” but it’s still the difference of squares.

EXAMPLE 7.50

Factor: $100 - h^2$.

Solution

	$100 - h^2$
Is this a diffe ence of squares? Yes.	$(10)^2 - (h)^2$
Factor as the product of conjugates.	$(10 - h)(10 + h)$

Check by multiplying.

$$(10 - h)(10 + h)$$
$$100 - h^2 \checkmark$$

Be careful not to rewrite the original expression as $h^2 - 100$.

Factor $h^2 - 100$ on your own and then notice how the result differs from $(10 - h)(10 + h)$.

TRY IT : : 7.99 Factor: $144 - x^2$.

TRY IT : : 7.100 Factor: $169 - p^2$.

To completely factor the binomial in the next example, we’ll factor a difference of squares twice!

EXAMPLE 7.51

Factor: $x^4 - y^4$.

Solution

	$x^4 - y^4$
Is this a diffe ence of squares? Yes.	$\left(x^2\right)^2 - \left(y^2\right)^2$
Factor it as the product of conjugates.	$\left(x^2 - y^2\right)\left(x^2 + y^2\right)$
Notice the fir t binomial is also a diffe ence of squares!	$\left((x)^2 - (y)^2\right)\left(x^2 + y^2\right)$
Factor it as the product of conjugates. The last factor, the sum of squares, cannot be factored.	$(x - y)(x + y)\left(x^2 + y^2\right)$

Check by multiplying.

$$(x - y)(x + y)\left(x^2 + y^2\right)$$
$$[(x - y)(x + y)]\left(x^2 + y^2\right)$$
$$\left(x^2 - y^2\right)\left(x^2 + y^2\right)$$
$$x^4 - y^4 \checkmark$$

TRY IT : : 7.101 Factor: $a^4 - b^4$.

TRY IT : : 7.102 Factor: $x^4 - 16$.

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may "disguise" the difference of squares and you won't recognize the perfect squares until you factor the GCF.

EXAMPLE 7.52

Factor: $8x^2y - 18y$.

Solution

	$8x^2y - 98y$
Is there a GCF? Yes, $2y$—factor it out!	$2y\left(4x^2 - 49\right)$
Is the binomial a diffe ence of squares? Yes.	$2y\left((2x)^2 - (7)^2\right)$
Factor as a product of conjugates.	$2y(2x - 7)(2x + 7)$

Check by multiplying.

$$2y(2x - 7)(2x + 7)$$
$$2y[(2x - 7)(2x + 7)]$$
$$2y\left(4x^2 - 49\right)$$
$$8x^2y - 98y \checkmark$$

TRY IT : : 7.103 Factor: $7xy^2 - 175x$.

TRY IT : : 7.104 Factor: $45a^2b - 80b$.

EXAMPLE 7.53

Factor: $6x^2 + 96$.

Solution

	$6x^2 + 96$
Is there a GCF? Yes, 6—factor it out!	$6\left(x^2 + 16\right)$
Is the binomial a diffe ence of squares? No, it is a sum of squares. Sums of squares do not factor!	

Check by multiplying.

$$6\left(x^2 + 16\right)$$
$$6x^2 + 96 \checkmark$$

TRY IT : : 7.105 Factor: $8a^2 + 200$.

TRY IT : : 7.106 Factor: $36y^2 + 81$.

Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

We'll check the first pattern and leave the second to you.

	$(a+b)(a^2 - ab + b^2)$
Distribute.	$a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$
Multiply.	$a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$
Combine like terms.	$a^3 + b^3$

Sum and Difference of Cubes Pattern

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

The two patterns look very similar, don't they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

$$a^3 \underbrace{+ b^3 = (a +}_{\text{same sign}} b)(a^2 - ab + b^2)$$
$$\underbrace{+ b^3 = (a + b)(a^2 -}_{\text{opposite signs}}$$

$$a^3 \underbrace{- b^3 = (a -}_{\text{same sign}} b)(a^2 + ab + b^2)$$
$$\underbrace{- b^3 = (a - b)(a^2 +}_{\text{opposite signs}}$$

The trinomial factor in the sum and difference of cubes pattern cannot be factored.

It can be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10 in Figure 7.3.

n	1	2	3	4	5	6	7	8	9	10
n^3	1	8	27	64	125	216	343	512	729	1000

Figure 7.3

EXAMPLE 7.54 HOW TO FACTOR THE SUM OR DIFFERENCE OF CUBES

Factor: $x^3 + 64$.

Solution

Step 1. Does the binomial fit the sum or difference of cubes pattern? • Is it a sum or difference? • Are the first and last terms perfect cubes?	This is a sum. Yes	$x^3 + 64$ $x^3 + 64$
Step 2. Write the terms as cubes.	Write them as x^3 and 4^3	$\underset{x^3 + 4^3}{a^3 + b^3}$
Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$\left(\underset{x+4}{a+b}\right)\left(\underset{x^2-4x+4^2}{a^2-ab+b^2}\right)$
Step 4. Simplify inside the parentheses.	Simplify 4^2.	$(x + 4)(x^2 - 4x + 16)$
Step 5. Check by multiplying the factors.		$\begin{array}{r} x^2 - 4x + 16 \\ x + 4 \\ \hline 4x^2 - 16x + 64 \\ x^3 - 4x^2 + 16x \\ \hline x^3 + 64 \checkmark \end{array}$

TRY IT : : 7.107 Factor: $x^3 + 27$.

TRY IT : : 7.108 Factor: $y^3 + 8$.

HOW TO : : FACTOR THE SUM OR DIFFERENCE OF CUBES.

To factor the sum or difference of cubes:

Step 1. Does the binomial fit the sum or difference of cubes pattern?
- Is it a sum or difference?
- Are the first and last terms perfect cubes?

Step 2. Write them as cubes.

Step 3. Use either the sum or difference of cubes pattern.

Step 4. Simplify inside the parentheses

Step 5. Check by multiplying the factors.

EXAMPLE 7.55

Factor: $x^3 - 1000$.

Solution

	$x^3 - 1000$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\underset{x^3 - 10^3}{a^3 - b^3}$

Use the difference of cubes pattern.	$\overset{a\ -\ b}{(x-10)}\overset{a^2\ +\ ab\ +\ b^2}{(x^2+10\cdot x+10^2)}$
Simplify.	$\overset{a\ -\ b}{(x-10)}\overset{a^2\ +\ ab\ +\ b^2}{(x^2+10x+100)}$
Check by multiplying.	

$(x-10)\ (x^2+10x+100)$

$$\begin{array}{r} x^2+10x+100 \\ x-100 \\ \hline x^3+10x^2+100x \\ -10x^2-100x-1000\ \checkmark \\ \hline x^3 \qquad\qquad -1000 \end{array}$$

TRY IT : : 7.109 Factor: u^3-125.

TRY IT : : 7.110 Factor: v^3-343.

Be careful to use the correct signs in the factors of the sum and difference of cubes.

EXAMPLE 7.56

Factor: $512-125p^3$.

Solution

	$512-125p^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3\ -\ b^3}{8^3-(5p)^3}$
Use the difference of cubes pattern.	$\overset{a\ -\ b}{(8-5p)}\overset{a^2\ +\ ab\ +\ b^2}{(8^2+8\cdot 5p+(5p)^2)}$
Simplify.	$\overset{a\ -\ b}{(8-5p)}\overset{a^2\ +\ ab\ +\ b^2}{(64+40p+25p^2)}$
Check by multiplying.	We'll leave the check to you.

TRY IT : : 7.111 Factor: $64-27x^3$.

TRY IT : : 7.112 Factor: $27-8y^3$.

EXAMPLE 7.57

Factor: $27u^3-125v^3$.

Solution

	$27u^3 - 125v^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(3u)^3} - \overset{b^3}{(5v)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{3u} - \overset{b}{5v}\right)\left(\overset{a^2}{(3u)^2} + \overset{ab}{3u \cdot 5v} + \overset{b^2}{(5v)^2}\right)$
Simplify.	$\left(\overset{a}{3u} - \overset{b}{5v}\right)\left(\overset{a^2}{9u^2} + \overset{ab}{15uv} + \overset{b^2}{25v^2}\right)$
Check by multiplying.	We'll leave the check to you.

> **TRY IT : : 7.113** Factor: $8x^3 - 27y^3$.

> **TRY IT : : 7.114** Factor: $1000m^3 - 125n^3$.

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

EXAMPLE 7.58

Factor: $5m^3 + 40n^3$.

Solution

	$5m^3 + 40n^3$
Factor the common factor.	$5(m^3 + 8n^3)$
This binomial is a sum. The first and last terms are perfect cubes.	
Write the terms as cubes.	$5\left(\overset{a^3}{m^3} + \overset{b^3}{(2n)^3}\right)$
Use the sum of cubes pattern.	$5\left(\overset{a}{m} + \overset{b}{2n}\right)\left(\overset{a^2}{m^2} - \overset{ab}{m \cdot 2n} + \overset{b^2}{(2n)^2}\right)$
Simplify.	$5\left(\overset{a}{m} + \overset{b}{2n}\right)\left(\overset{a^2}{m^2} - \overset{ab}{2m\,n} + \overset{b^2}{4n^2}\right)$

Check. To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by 5. We'll leave the multiplication for you.

$$5(m + 2n)\left(m^2 - 2mn + 4n^2\right)$$

> **TRY IT : : 7.115** Factor: $500p^3 + 4q^3$.

> **TRY IT : : 7.116** Factor: $432c^3 + 686d^3$.

MEDIA : :

Access these online resources for additional instruction and practice with factoring special products.

- **Sum of Difference of Cubes (https://openstax.org/l/25SumCubes)**
- **Difference of Cubes Factoring (https://openstax.org/l/25DiffCubes)**

7.4 EXERCISES

Practice Makes Perfect

Factor Perfect Square Trinomials

In the following exercises, factor.

215. $16y^2 + 24y + 9$

216. $25v^2 + 20v + 4$

217. $36s^2 + 84s + 49$

218. $49s^2 + 154s + 121$

219. $100x^2 - 20x + 1$

220. $64z^2 - 16z + 1$

221. $25n^2 - 120n + 144$

222. $4p^2 - 52p + 169$

223. $49x^2 - 28xy + 4y^2$

224. $25r^2 - 60rs + 36s^2$

225. $25n^2 + 25n + 4$

226. $100y^2 - 52y + 1$

227. $64m^2 - 34m + 1$

228. $100x^2 - 25x + 1$

229. $10k^2 + 80k + 160$

230. $64x^2 - 96x + 36$

231. $75u^3 - 30u^2v + 3uv^2$

232. $90p^3 + 300p^2q + 250pq^2$

Factor Differences of Squares

In the following exercises, factor.

233. $x^2 - 16$

234. $n^2 - 9$

235. $25v^2 - 1$

236. $169q^2 - 1$

237. $121x^2 - 144y^2$

238. $49x^2 - 81y^2$

239. $169c^2 - 36d^2$

240. $36p^2 - 49q^2$

241. $4 - 49x^2$

242. $121 - 25s^2$

243. $16z^4 - 1$

244. $m^4 - n^4$

245. $5q^2 - 45$

246. $98r^3 - 72r$

247. $24p^2 + 54$

248. $20b^2 + 140$

Factor Sums and Differences of Cubes

In the following exercises, factor.

249. $x^3 + 125$

250. $n^3 + 512$

251. $z^3 - 27$

252. $v^3 - 216$

253. $8 - 343t^3$

254. $125 - 27w^3$

255. $8y^3 - 125z^3$

256. $27x^3 - 64y^3$

257. $7k^3 + 56$

258. $6x^3 - 48y^3$

259. $2 - 16y^3$

260. $-2x^3 - 16y^3$

Mixed Practice

In the following exercises, factor.

261. $64a^2 - 25$

262. $121x^2 - 144$

263. $27q^2 - 3$

264. $4p^2 - 100$

265. $16x^2 - 72x + 81$

266. $36y^2 + 12y + 1$

267. $8p^2 + 2$

268. $81x^2 + 169$

269. $125 - 8y^3$

270. $27u^3 + 1000$

271. $45n^2 + 60n + 20$

272. $48q^3 - 24q^2 + 3q$

Everyday Math

273. Landscaping Sue and Alan are planning to put a 15 foot square swimming pool in their backyard. They will surround the pool with a tiled deck, the same width on all sides. If the width of the deck is *w*, the total area of the pool and deck is given by the trinomial $4w^2 + 60w + 225$. Factor the trinomial.

274. Home repair The height a twelve foot ladder can reach up the side of a building if the ladder's base is *b* feet from the building is the square root of the binomial $144 - b^2$. Factor the binomial.

Writing Exercises

275. Why was it important to practice using the binomial squares pattern in the chapter on multiplying polynomials?

276. How do you recognize the binomial squares pattern?

277. Explain why $n^2 + 25 \neq (n + 5)^2$. Use algebra, words, or pictures.

278. Maribel factored $y^2 - 30y + 81$ as $(y - 9)^2$. Was she right or wrong? How do you know?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
factor perfect square trinomials.			
factor differences of squares.			
factor sums and differences of cubes.			

ⓑ *On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

7.5 General Strategy for Factoring Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Recognize and use the appropriate method to factor a polynomial completely

Be Prepared!

Before you get started, take this readiness quiz.

1. Factor $y^2 - 2y - 24$.
   If you missed this problem, review **Example 7.23**.
2. Factor $3t^2 + 17t + 10$.
   If you missed this problem, review **Example 7.38**.
3. Factor $36p^2 - 60p + 25$.
   If you missed this problem, review **Example 7.42**.
4. Factor $5x^2 - 80$.
   If you missed this problem, review **Example 7.52**.

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. (In your next algebra course, more methods will be added to your repertoire.) The figure below summarizes all the factoring methods we have covered. **Factor polynomials.** outlines a strategy you should use when factoring polynomials.

General Strategy for Factoring Polynomials

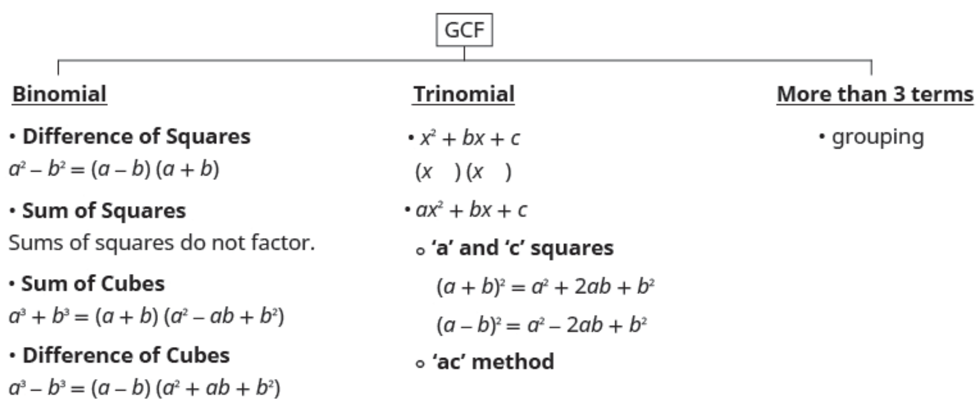


Figure 7.4

HOW TO : : FACTOR POLYNOMIALS.

Step 1. Is there a greatest common factor?
- Factor it out.

Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?
- If it is a binomial:
  Is it a sum?
  - Of squares? Sums of squares do not factor.
  - Of cubes? Use the sum of cubes pattern.

  Is it a difference?
  - Of squares? Factor as the product of conjugates.
  - Of cubes? Use the difference of cubes pattern.
- If it is a trinomial:
  Is it of the form $x^2 + bx + c$? Undo FOIL.
  Is it of the form $ax^2 + bx + c$?
  - If a and c are squares, check if it fits the trinomial square pattern.
  - Use the trial and error or "ac" method.
- If it has more than three terms:
  Use the grouping method.

Step 3. Check.
- Is it factored completely?
- Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are prime!

EXAMPLE 7.59

Factor completely: $4x^5 + 12x^4$.

Solution

Is there a GCF?	Yes, $4x^4$.	$4x^5 + 12x^4$
	Factor out the GCF.	$4x^4(x+3)$
In the parentheses, is it a binomial, a trinomial, or are there more than three terms?	Binomial.	
Is it a sum?		Yes.
Of squares? Of cubes?		No.
Check.		
Is the expression factored completely?		Yes.
Multiply.		

$$4x^4(x+3)$$
$$4x^4 \cdot x + 4x^4 \cdot 3$$
$$4x^5 + 12x^4 \checkmark$$

TRY IT : : 7.117 Factor completely: $3a^4 + 18a^3$.

TRY IT : : 7.118 Factor completely: $45b^6 + 27b^5$.

EXAMPLE 7.60

Factor completely: $12x^2 - 11x + 2$.

Solution

	$12x^2 - 11x + 2$
Is there a GCF?	No.
Is it a binomial, trinomial, or are there more than three terms?	Trinomial.
Are a and c perfect squares?	No, $a = 12$, not a perfect square.
Use trial and error or the "ac" method. We will use trial and error here.	$12x^2 - 11x + 2$ $1x, 12x$ $-1, -2$ $2x, 6x$ $3x, 4x$

$12x^2 - 11x + 2$	
Possible factors	**Product**
$(x - 1)(12x - 2)$	Not an option
$(x - 2)(12x - 1)$	$12x^2 - 25x + 2$
$(2x - 1)(6x - 2)$	Not an option
$(2x - 2)(6x - 1)$	Not an option
$(3x - 1)(4x - 2)$	Not an option
$(3x - 2)(4x - 1)$	$12x^2 - 11x + 2$

If the trinomial has no common factors, then neither factor can contain a common factor. That means each of these combinations is not an option.

Check.

$$(3x - 2)(4x - 1)$$

$$12x^2 - 3x - 8x + 2$$

$$12x^2 - 11x + 2 \checkmark$$

TRY IT : : 7.119 Factor completely: $10a^2 - 17a + 6$.

TRY IT : : 7.120 Factor completely: $8x^2 - 18x + 9$.

EXAMPLE 7.61

Factor completely: $g^3 + 25g$.

Solution

Is there a GCF?	Yes, g.	$g^3 + 25g$
Factor out the GCF.		$g(g^2 + 25)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum ? Of squares?	Yes.	Sums of squares are prime.
Check.		
Is the expression factored completely?	Yes.	

Multiply.

$g(g^2 + 25)$

$g^3 + 25g$ ✓

TRY IT : : 7.121 Factor completely: $x^3 + 36x$.

TRY IT : : 7.122 Factor completely: $27y^2 + 48$.

EXAMPLE 7.62

Factor completely: $12y^2 - 75$.

Solution

Is there a GCF?	Yes, 3.	$12y^2 - 75$
Factor out the GCF.		$3(4y^2 - 25)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum?	No.	
Is it a diffe ence? Of squares or cubes?	Yes, squares.	$3((2y)^2 - (5)^2)$
Write as a product of conjugates.		$3(2y - 5)(2y + 5)$
Check.		
Is the expression factored completely?	Yes.	

Neither binomial is a diffe ence of squares.

Multiply.

$3(2y - 5)(2y + 5)$

$3(4y^2 - 25)$

$12y^2 - 75$ ✓

TRY IT : : 7.123 Factor completely: $16x^3 - 36x$.

TRY IT : : 7.124 Factor completely: $27y^2 - 48$.

EXAMPLE 7.63

Factor completely: $4a^2 - 12ab + 9b^2$.

Solution

Is there a GCF?	No.	$4a^2 - 12ab + 9b^2$
Is it a binomial, trinomial, or are there more terms?		
Trinomial with $a \neq 1$. But the first term is a perfect square.		
Is the last term a perfect square?	Yes.	$(2a)^2 - 12ab + (3b)^2$
Does it fit the pattern, $a^2 - 2ab + b^2$?	Yes.	$(2a)^2 - \underbrace{12ab}_{\substack{-2(2a)(3b)\\12ab}} + (3b)^2$
Write it as a square.		$(2a - 3b)^2$
Check your answer.		
Is the expression factored completely?		
Yes.		
The binomial is not a difference of squares.		
Multiply.		
$(2a - 3b)^2$		
$(2a)^2 - 2 \cdot 2a \cdot 3b + (3b)^2$		
$4a^2 - 12ab + 9b^2$ ✓		

TRY IT : : 7.125 Factor completely: $4x^2 + 20xy + 25y^2$.

TRY IT : : 7.126 Factor completely: $9m^2 + 42mn + 49n^2$.

EXAMPLE 7.64

Factor completely: $6y^2 - 18y - 60$.

Solution

Is there a GCF?	Yes, 6.	$6y^2 - 18y - 60$
Factor out the GCF.	Trinomial with leading coefficient	$6\left(y^2 - 3y - 10\right)$
In the parentheses, is it a binomial, trinomial, or are there more terms?		
“Undo” FOIL.	$6(y \quad)(y \quad)$	$6(y + 2)(y - 5)$
Check your answer.		
Is the expression factored completely?		Yes.
Neither binomial is a diffe ence of squares.		
Multiply.		

$$6(y + 2)(y - 5)$$
$$6\left(y^2 - 5y + 2y - 10\right)$$
$$6\left(y^2 - 3y - 10\right)$$
$$6y^2 - 18y - 60 \checkmark$$

TRY IT : : 7.127 Factor completely: $8y^2 + 16y - 24$.

TRY IT : : 7.128 Factor completely: $5u^2 - 15u - 270$.

EXAMPLE 7.65

Factor completely: $24x^3 + 81$.

Solution

Is there a GCF?	Yes, 3.	$24x^3 + 81$
Factor it out.		$3\left(8x^3 + 27\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum or difference?	Sum.	
Of squares or cubes?	Sum of cubes.	$3\left(\overset{a^3}{(2x)^3} \overset{+}{+} \overset{b^3}{(3)^3}\right)$
Write it using the sum of cubes pattern.		$3\left(\overset{a}{2x} \overset{+}{+} \overset{b}{3}\right)\left(\overset{a^2}{(2x)^2} \overset{-}{-} \overset{ab}{2x \cdot 3} \overset{+}{+} \overset{b^2}{3^2}\right)$
Is the expression factored completely?	Yes.	$3(2x + 3)\left(4x^2 - 6x + 9\right)$
Check by multiplying.		We leave the check to you.

TRY IT : : 7.129 Factor completely: $250m^3 + 432$.

TRY IT : : 7.130 Factor completely: $81q^3 + 192$.

EXAMPLE 7.66

Factor completely: $2x^4 - 32$.

Solution

Is there a GCF?	Yes, 2.	$2x^4 - 32$
Factor it out.		$2(x^4 - 16)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum or diffe ence?	Yes.	
Of squares or cubes?	Diffe ence of squares.	$2\left(\left(x^2\right)^2 - (4)^2\right)$
Write it as a product of conjugates.		$2(x^2 - 4)(x^2 + 4)$
The fir t binomial is again a diffe ence of squares.		$2\left((x)^2 - (2)^2\right)(x^2 + 4)$
Write it as a product of conjugates.		$2(x - 2)(x + 2)(x^2 + 4)$
Is the expression factored completely?	Yes.	
None of these binomials is a diffe ence of squares.		

Check your answer.

Multiply.

$2(x - 2)(x + 2)(x^2 + 4)$

$2(x^2 - 4)(x^2 + 4)$

$2(x^4 - 16)$

$2x^4 - 32$ ✓

TRY IT : : 7.131 Factor completely: $4a^4 - 64$.

TRY IT : : 7.132 Factor completely: $7y^4 - 7$.

EXAMPLE 7.67

Factor completely: $3x^2 + 6bx - 3ax - 6ab$.

Solution

Is there a GCF?	Yes, 3.	$3x^2 + 6bx - 3ax - 6ab$
Factor out the GCF.		$3\left(x^2 + 2bx - ax - 2ab\right)$
In the parentheses, is it a binomial, trinomial, or are there more terms?	More than 3 terms.	
Use grouping.		$3[x(x + 2b) - a(x + 2b)]$ $3(x + 2b)(x - a)$

Check your answer.

Is the expression factored completely? Yes.
Multiply.

$$3(x + 2b)(x - a)$$
$$3\left(x^2 - ax + 2bx - 2ab\right)$$
$$3x^2 - 3ax + 6bx - 6ab \checkmark$$

TRY IT : : 7.133 Factor completely: $6x^2 - 12xc + 6bx - 12bc$.

TRY IT : : 7.134 Factor completely: $16x^2 + 24xy - 4x - 6y$.

EXAMPLE 7.68

Factor completely: $10x^2 - 34x - 24$.

Solution

Is there a GCF?	Yes, 2.	$10x^2 - 34x - 24$
Factor out the GCF.		$2\left(5x^2 - 17x - 12\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Trinomial with $a \neq 1$.	
Use trial and error or the “ac” method.		$2(5x^2 - 17x - 12)$ $2(5x + 3)(x - 4)$

Check your answer. Is the expression factored completely? Yes.

Multiply.

$$2(5x + 3)(x - 4)$$
$$2\left(5x^2 - 20x + 3x - 12\right)$$
$$2\left(5x^2 - 17x - 12\right)$$
$$10x^2 - 34x - 24 \checkmark$$

TRY IT : : 7.135 Factor completely: $4p^2 - 16p + 12$.

TRY IT : : 7.136 Factor completely: $6q^2 - 9q - 6$.

7.5 EXERCISES

Practice Makes Perfect

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

279. $10x^4 + 35x^3$

280. $18p^6 + 24p^3$

281. $y^2 + 10y - 39$

282. $b^2 - 17b + 60$

283. $2n^2 + 13n - 7$

284. $8x^2 - 9x - 3$

285. $a^5 + 9a^3$

286. $75m^3 + 12m$

287. $121r^2 - s^2$

288. $49b^2 - 36a^2$

289. $8m^2 - 32$

290. $36q^2 - 100$

291. $25w^2 - 60w + 36$

292. $49b^2 - 112b + 64$

293. $m^2 + 14mn + 49n^2$

294. $64x^2 + 16xy + y^2$

295. $7b^2 + 7b - 42$

296. $3n^2 + 30n + 72$

297. $3x^3 - 81$

298. $5t^3 - 40$

299. $k^4 - 16$

300. $m^4 - 81$

301. $15pq - 15p + 12q - 12$

302. $12ab - 6a + 10b - 5$

303. $4x^2 + 40x + 84$

304. $5q^2 - 15q - 90$

305. $u^5 + u^2$

306. $5n^3 + 320$

307. $4c^2 + 20cd + 81d^2$

308. $25x^2 + 35xy + 49y^2$

309. $10m^4 - 6250$

310. $3v^4 - 768$

Everyday Math

311. Watermelon drop A springtime tradition at the University of California San Diego is the Watermelon Drop, where a watermelon is dropped from the seventh story of Urey Hall.

ⓐ The binomial $-16t^2 + 80$ gives the height of the watermelon t seconds after it is dropped. Factor the greatest common factor from this binomial.

ⓑ If the watermelon is thrown down with initial velocity 8 feet per second, its height after t seconds is given by the trinomial $-16t^2 - 8t + 80$. Completely factor this trinomial.

312. Pumpkin drop A fall tradition at the University of California San Diego is the Pumpkin Drop, where a pumpkin is dropped from the eleventh story of Tioga Hall.

ⓐ The binomial $-16t^2 + 128$ gives the height of the pumpkin t seconds after it is dropped. Factor the greatest common factor from this binomial.

ⓑ If the pumpkin is thrown down with initial velocity 32 feet per second, its height after t seconds is given by the trinomial $-16t^2 - 32t + 128$. Completely factor this trinomial.

Writing Exercises

313. The difference of squares $y^4 - 625$ can be factored as $\left(y^2 - 25\right)\left(y^2 + 25\right)$. But it is not *completely* factored. What more must be done to completely factor it?

314. Of all the factoring methods covered in this chapter (GCF, grouping, undo FOIL, ‘ac’ method, special products) which is the easiest for you? Which is the hardest? Explain your answers.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don’t get it!
recognize and use the appropriate method to factor a polynomial completely.			

ⓑ *Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

7.6 Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations by using the Zero Product Property
- Solve quadratic equations factoring
- Solve applications modeled by quadratic equations

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $5y - 3 = 0$.
   If you missed this problem, review **Example 2.27**.
2. Solve: $10a = 0$.
   If you missed this problem, review **Example 2.13**.
3. Combine like terms: $12x^2 - 6x + 4x$.
   If you missed this problem, review **Example 1.24**.
4. Factor $n^3 - 9n^2 - 22n$ completely.
   If you missed this problem, review **Example 7.32**.

We have already solved linear equations, equations of the form $ax + by = c$. In linear equations, the variables have no exponents. Quadratic equations are equations in which the variable is squared. Listed below are some examples of quadratic equations:

$$x^2 + 5x + 6 = 0 \qquad 3y^2 + 4y = 10 \qquad 64u^2 - 81 = 0 \qquad n(n+1) = 42$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get $n^2 + n$.

The general form of a quadratic equation is $ax^2 + bx + c = 0, \text{ with } a \neq 0$.

Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation.

$$a,\ b,\ \text{ and } c \text{ are real numbers and } a \neq 0$$

To solve quadratic equations we need methods different than the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Solve Quadratic Equations Using the Zero Product Property

We will first solve some quadratic equations by using the Zero Product Property. The **Zero Product Property** says that if the product of two quantities is zero, it must be that at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

Zero Product Property

If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.

We will now use the Zero Product Property, to solve a quadratic equation.

EXAMPLE 7.69 HOW TO USE THE ZERO PRODUCT PROPERTY TO SOLVE A QUADRATIC EQUATION

Solve: $(x+1)(x-4) = 0$.

Solution

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(x+1)(x-4)=0$ $x+1=0 \quad \text{or} \quad x-4=0$
Step 2. Solve the linear equations.	Solve each equation.	$x=-1 \quad \text{or} \quad x=4$
Step 3. Check.	Substitute each solution separately into the original equation.	$x=-1$ $(x+1)(x-4)=0$ $(-1+1)(-1-4)\stackrel{?}{=}0$ $(0)(-5)\stackrel{?}{=}0$ $0=0\checkmark$ $x=4$ $(x+1)(x-4)=0$ $(4+1)(4-4)\stackrel{?}{=}0$ $(5)(0)\stackrel{?}{=}0$ $0=0\checkmark$

TRY IT : : 7.137 Solve: $(x-3)(x+5)=0$.

TRY IT : : 7.138 Solve: $(y-6)(y+9)=0$.

We usually will do a little more work than we did in this last example to solve the linear equations that result from using the Zero Product Property.

EXAMPLE 7.70

Solve: $(5n-2)(6n-1)=0$.

Solution

		$(5n-2)(6n-1)=0$
Use the Zero Product Property to set each factor to 0.	$5n-2=0$	$6n-1=0$
Solve the equations.	$n=\frac{2}{5}$	$n=\frac{1}{6}$
Check your answers.		

$n = \frac{2}{5}$	$n = \frac{1}{6}$
$(5n-2)(6n-1) = 0$	$(5n-2)(6n-1) = 0$
$\left(5 \cdot \frac{2}{5} - 2\right)\left(6 \cdot \frac{2}{5} - 1\right) \stackrel{?}{=} 0$	$\left(5 \cdot \frac{1}{6} - 2\right)\left(6 \cdot \frac{1}{6} - 1\right) \stackrel{?}{=} 0$
$(2-2)\left(\frac{12}{5} - \frac{5}{5}\right) \stackrel{?}{=} 0$	$\left(\frac{5}{6} - \frac{12}{6}\right)(1-1) \stackrel{?}{=} 0$
$(0)\left(\frac{7}{5}\right) \stackrel{?}{=} 0$	$\left(-\frac{7}{6}\right)(0) \stackrel{?}{=} 0$
$0 = 0 \checkmark$	$0 = 0 \checkmark$

TRY IT : : 7.139 Solve: $(3m-2)(2m+1) = 0$.

TRY IT : : 7.140 Solve: $(4p+3)(4p-3) = 0$.

Notice when we checked the solutions that each of them made just one factor equal to zero. But the product was zero for both solutions.

EXAMPLE 7.71

Solve: $3p(10p+7) = 0$.

Solution

		$3p(10p+7) = 0$
Use the Zero Product Property to set each factor to 0.	$3p = 0$	$10p + 7 = 0$
Solve the equations.	$p = 0$	$10p = -7$
		$p = -\frac{7}{10}$
Check your answers.		

$p = 0$	$p = -\frac{7}{10}$
$3p(10p+7) = 0$	$3p(10p+7) = 0$
$3 \cdot 0(10 \cdot 0 + 7) \stackrel{?}{=} 0$	$3\left(-\frac{7}{10}\right)10\left(-\frac{7}{10}\right) + 7 \stackrel{?}{=} 0$
$0(0+7) \stackrel{?}{=} 0$	$\left(-\frac{21}{10}\right)(-7+7) \stackrel{?}{=} 0$
$0(7) \stackrel{?}{=} 0$	$\left(-\frac{21}{10}\right)(0) \stackrel{?}{=} 0$
$0 = 0 \checkmark$	$0 = 0 \checkmark$

TRY IT : : 7.141 Solve: $2u(5u-1) = 0$.

TRY IT : : 7.142 Solve: $w(2w + 3) = 0$.

It may appear that there is only one factor in the next example. Remember, however, that $(y - 8)^2$ means $(y - 8)(y - 8)$.

EXAMPLE 7.72

Solve: $(y - 8)^2 = 0$.

Solution

		$(y - 8)^2 = 0$
Rewrite the left side as a product.		$(y - 8)(y - 8) = 0$
Use the Zero Product Property and set each factor to 0.	$y - 8 = 0$	$y - 8 = 0$
Solve the equations.	$y = 8$	$y = 8$
When a solution repeats, we call it a double root.		
Check your answer.		

$$\begin{aligned} y &= 8 \\ (y - 8)^2 &= 0 \\ (8 - 8)^2 &\stackrel{?}{=} 0 \\ (0)^2 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$$

TRY IT : : 7.143 Solve: $(x + 1)^2 = 0$.

TRY IT : : 7.144 Solve: $(v - 2)^2 = 0$.

Solve Quadratic Equations by Factoring

Each of the equations we have solved in this section so far had one side in factored form. In order to use the Zero Product Property, the quadratic equation must be factored, with zero on one side. So we be sure to start with the quadratic equation in standard form, $ax^2 + bx + c = 0$. Then we factor the expression on the left.

EXAMPLE 7.73 HOW TO SOLVE A QUADRATIC EQUATION BY FACTORING

Solve: $x^2 + 2x - 8 = 0$.

Solution

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.	The equation is already in standard form.	$x^2 + 2x - 8 = 0$
Step 2. Factor the quadratic expression.	Factor $x^2 + 2x - 8$ $(x + 4)(x - 2)$	$(x + 4)(x - 2) = 0$

Step 3. Use the Zero Product Property.	Set each factor equal to zero.	$x+4=0$ or $x-2=0$
Step 4. Solve the linear equations.	We have two linear equations.	$x=-4$ or $x=2$
Step 5. Check.	Substitute each solution separately into the original equation.	$x^2+2x-8=0$ $x=-4$ $(-4)^2-2(-4)-8\stackrel{?}{=}0$ $16+(-8)-8\stackrel{?}{=}0$ $0=0$ ✓ $x^2+2x-8=0$ $x=2$ $2^2-2(2)-8\stackrel{?}{=}0$ $4+4-8\stackrel{?}{=}0$ $0=0$ ✓

TRY IT : : 7.145 Solve: $x^2-x-12=0$.

TRY IT : : 7.146 Solve: $b^2+9b+14=0$.

HOW TO : : SOLVE A QUADRATIC EQUATION BY FACTORING.

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.

Step 2. Factor the quadratic expression.

Step 3. Use the Zero Product Property.

Step 4. Solve the linear equations.

Step 5. Check.

Before we factor, we must make sure the quadratic equation is in standard form.

EXAMPLE 7.74

Solve: $2y^2=13y+45$.

Solution

	$2y^2=13y+45$	
Write the quadratic equation in standard form.	$2y^2-13y-45=0$	
Factor the quadratic expression.	$(2y+5)(y-9)=0$	
Use the Zero Product Property to set each factor to 0.	$2y+5=0$	$y-9=0$
Solve each equation.	$y=-\frac{5}{2}$	$y=9$

Check your answers.	
$y = -\frac{5}{2}$	$y = 9$
$2y^2 = 13y + 45$	$2y^2 = 13y + 45$
$2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right) + 45$	$2(9)^2 \stackrel{?}{=} 13(9) + 45$
$2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right) + \frac{90}{2}$	$2(81) \stackrel{?}{=} 117 + 45$
$\frac{25}{2} = \frac{25}{2}$ ✓	$162 = 162$ ✓

TRY IT : : 7.147 Solve: $3c^2 = 10c - 8$.

TRY IT : : 7.148 Solve: $2d^2 - 5d = 3$.

EXAMPLE 7.75

Solve: $5x^2 - 13x = 7x$.

Solution

	$5x^2 - 13x = 7x$	
Write the quadratic equation in standard form.	$5x^2 - 20x = 0$	
Factor the left side of the equation.	$5x(x - 4) = 0$	
Use the Zero Product Property to set each factor to 0.	$5x = 0$	$x - 4 = 0$
Solve each equation.	$x = 0$	$x = 4$
Check your answers.		

$x = 0$	$x = 4$
$5x^2 - 13x = 7x$	$5x^2 - 13x = 7x$
$5(0)^2 - 13(0) \stackrel{?}{=} 7(0)$	$5(4)^2 - 13(4) \stackrel{?}{=} 7(4)$
$0 - 0 \stackrel{?}{=} 0$	$5(16) - 52 \stackrel{?}{=} 28$
$0 = 0$ ✓	$28 = 28$ ✓

TRY IT : : 7.149 Solve: $6a^2 + 9a = 3a$.

TRY IT : : 7.150 Solve: $45b^2 - 2b = -17b$.

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

EXAMPLE 7.76

Solve: $144q^2 = 25$.

Solution

	$144q^2 = 25$
Write the quadratic equation in standard form.	$144q^2 - 25 = 0$
Factor. It is a diffe ence of squares.	$(12q - 5)(12q + 5) = 0$
Use the Zero Product Property to set each factor to 0.	$12q - 5 = 0 \qquad 12q + 5 = 0$
Solve each equation.	$12q = 5 \qquad 12q = -5$ $q = \frac{5}{12} \qquad q = -\frac{5}{12}$
Check your answers.	

TRY IT : : 7.151 Solve: $25p^2 = 49$.

TRY IT : : 7.152 Solve: $36x^2 = 121$.

The left side in the next example is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

EXAMPLE 7.77

Solve: $(3x - 8)(x - 1) = 3x$.

Solution

	$(3x - 8)(x - 1) = 3x$
Multiply the binomials.	$3x^2 - 11x + 8 = 3x$
Write the quadratic equation in standard form.	$3x^2 - 14x + 8 = 0$
Factor the trinomial.	$(3x - 2)(x - 4) = 0$
Use the Zero Product Property to set each factor to 0.	$3x - 2 = 0 \qquad x - 4 = 0$
Solve each equation.	$3x = 2 \qquad x = 4$ $x = \frac{2}{3}$
Check your answers.	The check is left to you!

TRY IT : : 7.153 Solve: $(2m + 1)(m + 3) = 12m$.

TRY IT : : 7.154 Solve: $(k+1)(k-1)=8$.

The Zero Product Property also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree more than two by using the Zero Product Property, just like we solved quadratic equations.

EXAMPLE 7.78

Solve: $9m^3+100m=60m^2$.

Solution

	$9m^3+100m = 60m^2$
Bring all the terms to one side so that the other side is zero.	$9m^3-60m^2+100m = 0$
Factor the greatest common factor fir t.	$m\left(9m^2-60m+100\right) = 0$
Factor the trinomial.	$m(3m-10)(3m-10) = 0$
Use the Zero Product Property to set each factor to 0.	$m=0 \quad 3m-10=0 \quad 3m-10=0$
Solve each equation.	$m=0 \quad m=\frac{10}{3} \quad m=\frac{10}{3}$
Check your answers.	The check is left to you.

TRY IT : : 7.155 Solve: $8x^3=24x^2-18x$.

TRY IT : : 7.156 Solve: $16y^2=32y^3+2y$.

When we factor the quadratic equation in the next example we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

EXAMPLE 7.79

Solve: $4x^2=16x+84$.

Solution

	$4x^2 = 16x+84$
Write the quadratic equation in standard form.	$4x^2-16x-84 = 0$
Factor the greatest common factor fir t.	$4\left(x^2-4x-21\right) = 0$
Factor the trinomial.	$4(x-7)(x+3) = 0$
Use the Zero Product Property to set each factor to 0.	$4\neq 0 \quad x-7=0 \quad x+3=0$
Solve each equation.	$4\neq 0 \quad x=7 \quad x=-3$
Check your answers.	The check is left to you.

TRY IT : : 7.157 Solve: $18a^2-30=-33a$.

TRY IT : : 7.158 Solve: $123b = -6 - 60b^2$.

Solve Applications Modeled by Quadratic Equations

The problem solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to quadratic equations. We will copy the problem solving strategy here so we can use it for reference.

HOW TO : : USE A PROBLEM-SOLVING STRATEGY TO SOLVE WORD PROBLEMS.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.
Step 2. **Identify** what we are looking for.
Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.
Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
Step 5. **Solve** the equation using good algebra techniques.
Step 6. **Check** the answer in the problem and make sure it makes sense.
Step 7. **Answer** the question with a complete sentence.

We will start with a number problem to get practice translating words into a quadratic equation.

EXAMPLE 7.80

The product of two consecutive integers is 132. Find the integers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive integers.
Step 3. Name what we are looking for.	Let $n =$ the fir t integer $n + 1 =$ the next consecutive integer
Step 4. Translate into an equation. Restate the problem in a sentence.	The product of the two consecutive integers is 132.
	The fir t integer times the next integer is 132.
Translate to an equation.	$n(n+1) = 132$
Step 5. Solve the equation.	$n^2 + n = 132$
Bring all the terms to one side.	$n^2 + n - 132 = 0$
Factor the trinomial.	$(n-11)(n+12) = 0$
Use the zero product property.	$n - 11 = 0 \qquad n + 12 = 0$
Solve the equations.	$n = 11 \qquad n = -12$

There are two values for n that are solutions to this problem. So there are two sets of consecutive integers that will work.

If the fir t integer is $n = 11$	If the fir t integer is $n = -12$
then the next integer is $n + 1$	then the next integer is $n + 1$
$11 + 1$	$-12 + 1$
12	-11

Step 6. Check the answer.

The consecutive integers are $11, 12$ and $-11, -12$. The product $11 \cdot 12 = 132$ and the product $-11(-12) = 132$. Both pairs of consecutive integers are solutions.

Step 7. Answer the question. The consecutive integers are $11, 12$ and $-11, -12$.

TRY IT : : 7.159 The product of two consecutive integers is 240. Find the integers.

TRY IT : : 7.160 The product of two consecutive integers is 420. Find the integers.

Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give 132.

In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

EXAMPLE 7.81

A rectangular garden has an area 15 square feet. The length of the garden is two feet more than the width. Find the length and width of the garden.

Solution

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.		
Step 2. Identify what you are looking for.	We are looking for the length and width.	
Step 3. Name what you are looking for. The length is two feet more than width.	Let W = the width of the garden. $W + 2$ = the length of the garden	
Step 4. Translate into an equation. Restate the important information in a sentence.	The area of the rectangular garden is 15 square feet.	
Use the formula for the area of a rectangle.	$A = L \cdot W$	
Substitute in the variables.	$15 = (W + 2)W$	
Step 5. Solve the equation. Distribute first.	$15 = W^2 + 2W$	
Get zero on one side.	$0 = W^2 + 2W - 15$	
Factor the trinomial.	$0 = (W + 5)(W - 3)$	
Use the Zero Product Property.	$0 = W + 5$	$0 = W - 3$
Solve each equation.	$-5 = W$	$3 = W$

Since *W* is the width of the garden, it does not make sense for it to be negative. We eliminate that value for *W*.	$\cancel{-5 = W}$ $W = 3$	$3 = W$ Width is 3 feet.
Find the value of the length.	$W + 2 = \text{length}$	
	$3 + 2$	
	5	Length is 5 feet.
Step 6. Check the answer. Does the answer make sense?		
 $A = L \cdot W$ $A = 3 \cdot 5$ $A = 15$		
	Yes, this makes sense.	
Step 7. Answer the question.	The width of the garden is 3 feet and the length is 5 feet.	

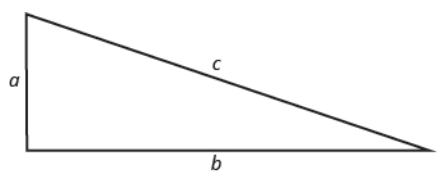
TRY IT : : 7.161

A rectangular sign has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

TRY IT : : 7.162

A rectangular patio has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

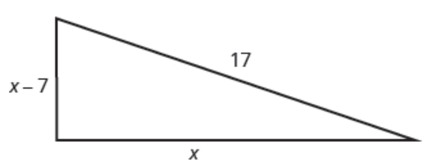
In an earlier chapter, we used the Pythagorean Theorem $\left(a^2 + b^2 = c^2\right)$. It gave the relation between the legs and the hypotenuse of a right triangle.



We will use this formula to in the next example.

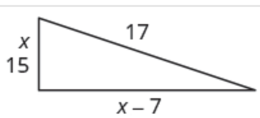
EXAMPLE 7.82

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle, as shown below. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the deck.



Solution

Step 1. Read the problem.			
Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the deck.		
Step 3. Name what you are looking for. One side is 7 less than the other.	Let x = length of a side of the deck $x - 7$ = length of other side		
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2 + b^2 = c^2$		
Substitute in the variables.	$x^2 + (x - 7)^2 = 17^2$		
Step 5. Solve the equation.	$x^2 + x^2 - 14x + 49 = 289$		
Simplify.	$2x^2 - 14x + 49 = 289$		
It is a quadratic equation, so get zero on one side.	$2x^2 - 14x - 240 = 0$		
Factor the greatest common factor.	$2(x^2 - 7x - 120) = 0$		
Factor the trinomial.	$2(x - 15)(x + 8) = 0$		
Use the Zero Product Property.	$2 \neq 0$	$x - 15 = 0$	$x + 8 = 0$
Solve.	$2 \neq 0$	$x = 15$	$x = -8$
Since x is a side of the triangle, $x = -8$ does not make sense.	$2 \neq 0$	$x = 15$	$\cancel{x = -8}$
Find the length of the other side.			
If the length of one side is	$x = 15$		
then the length of the other side is	$x - 7$		
	$15 - 7$		
	8 is the length of the other side.		
Step 6. Check the answer. Do these numbers make sense?			



$15 - 7$

8

$$a^2 + b^2 = c^2$$
$$15^2 + 8^2 \stackrel{?}{=} 17^2$$
$$225 + 64 \stackrel{?}{=} 289$$
$$289 = 289 \checkmark$$

Step 7. Answer the question.	The sides of the deck are 8, 15, and 17 feet.

TRY IT : : 7.163

A boat's sail is a right triangle. The length of one side of the sail is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the sail.

TRY IT : : 7.164

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of one of the other legs. Find the lengths of the hypotenuse and the other leg.

7.6 EXERCISES

Practice Makes Perfect

Use the Zero Product Property

In the following exercises, solve.

315. $(x-3)(x+7)=0$

316. $(y-11)(y+1)=0$

317. $(3a-10)(2a-7)=0$

318. $(5b+1)(6b+1)=0$

319. $6m(12m-5)=0$

320. $2x(6x-3)=0$

321. $(y-3)^2=0$

322. $(b+10)^2=0$

323. $(2x-1)^2=0$

324. $(3y+5)^2=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

325. $x^2+7x+12=0$

326. $y^2-8y+15=0$

327. $5a^2-26a=24$

328. $4b^2+7b=-3$

329. $4m^2=17m-15$

330. $n^2=5-6n \quad n^2=5n-6$

331. $7a^2+14a=7a$

332. $12b^2-15b=-9b$

333. $49m^2=144$

334. $625=x^2$

335. $(y-3)(y+2)=4y$

336. $(p-5)(p+3)=-7$

337. $(2x+1)(x-3)=-4x$

338. $(x+6)(x-3)=-8$

339. $16p^3=24p^2+9p$

340. $m^3-2m^2=-m$

341. $20x^2-60x=-45$

342. $3y^2-18y=-27$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

343. The product of two consecutive integers is 56. Find the integers.

344. The product of two consecutive integers is 42. Find the integers.

345. The area of a rectangular carpet is 28 square feet. The length is three feet more than the width. Find the length and the width of the carpet.

346. A rectangular retaining wall has area 15 square feet. The height of the wall is two feet less than its length. Find the height and the length of the wall.

347. A pennant is shaped like a right triangle, with hypotenuse 10 feet. The length of one side of the pennant is two feet longer than the length of the other side. Find the length of the two sides of the pennant.

348. A reflecting pool is shaped like a right triangle, with one leg along the wall of a building. The hypotenuse is 9 feet longer than the side along the building. The third side is 7 feet longer than the side along the building. Find the lengths of all three sides of the reflecting pool.

Mixed Practice

In the following exercises, solve.

349. $(x+8)(x-3)=0$

350. $(3y-5)(y+7)=0$

351. $p^2+12p+11=0$

352. $q^2 - 12q - 13 = 0$

353. $m^2 = 6m + 16$

354. $4n^2 + 19n = 5$

355. $a^3 - a^2 - 42a = 0$

356. $4b^2 - 60b + 224 = 0$

357. The product of two consecutive integers is 110. Find the integers.

358. The length of one leg of a right triangle is three more than the other leg. If the hypotenuse is 15, find the lengths of the two legs.

Everyday Math

359. Area of a patio If each side of a square patio is increased by 4 feet, the area of the patio would be 196 square feet. Solve the equation $(s+4)^2 = 196$ for *s* to find the length of a side of the patio.

360. Watermelon drop A watermelon is dropped from the tenth story of a building. Solve the equation $-16t^2 + 144 = 0$ for t to find the number of seconds it takes the watermelon to reach the ground.

Writing Exercises

361. Explain how you solve a quadratic equation. How many answers do you expect to get for a quadratic equation?

362. Give an example of a quadratic equation that has a GCF and none of the solutions to the equation is zero.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations by using the Zero Product Property.			
solve quadratic equations by factoring.			
solve applications modeled by quadratic equations.			

ⓑ *Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

CHAPTER 7 REVIEW

KEY TERMS

difference of squares pattern If *a* and *b* are real numbers,

$$a^2 - b^2 = (a-b)(a+b) \qquad a^2 \overset{\text{difference}}{-} b^2 = \underbrace{(a-b)(a+b)}_{\text{conjugates}}$$

squares

factoring Factoring is splitting a product into factors; in other words, it is the reverse process of multiplying.

greatest common factor The greatest common factor is the largest expression that is a factor of two or more expressions is the greatest common factor (GCF).

perfect square trinomials pattern If *a* and *b* are real numbers,

$$a^2 + 2ab + b^2 = (a+b)^2 \qquad a^2 - 2ab + b^2 = (a-b)^2$$

prime polynomials Polynomials that cannot be factored are prime polynomials.

quadratic equations are equations in which the variable is squared.

sum and difference of cubes pattern

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

Zero Product Property The Zero Product Property states that, if the product of two quantities is zero, at least one of the quantities is zero.

KEY CONCEPTS

7.1 Greatest Common Factor and Factor by Grouping

- **Finding the Greatest Common Factor (GCF):** To find the GCF of two expressions:
  Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
  Step 2. List all factors—matching common factors in a column. In each column, circle the common factors.
  Step 3. Bring down the common factors that all expressions share.
  Step 4. Multiply the factors as in **Example 7.2**.
- **Factor the Greatest Common Factor from a Polynomial:** To factor a greatest common factor from a polynomial:
  Step 1. Find the GCF of all the terms of the polynomial.
  Step 2. Rewrite each term as a product using the GCF.
  Step 3. Use the 'reverse' Distributive Property to factor the expression.
  Step 4. Check by multiplying the factors as in **Example 7.5**.
- **Factor by Grouping:** To factor a polynomial with 4 four or more terms
  Step 1. Group terms with common factors.
  Step 2. Factor out the common factor in each group.
  Step 3. Factor the common factor from the expression.
  Step 4. Check by multiplying the factors as in **Example 7.15**.

7.2 Factor Quadratic Trinomials with Leading Coefficient 1

- **Factor trinomials of the form $x^2 + bx + c$**
  Step 1. Write the factors as two binomials with first terms *x*: $(x \quad)(x \quad)$.
  Step 2. Find two numbers *m* and *n* that
  Multiply to *c*, $m \cdot n = c$
  Add to *b*, $m + n = b$

Step 3. Use *m* and *n* as the last terms of the factors: $(x+m)(x+n)$.

Step 4. Check by multiplying the factors.

7.3 Factor Quadratic Trinomials with Leading Coefficient Other than 1

- **Factor Trinomials of the Form ax^2+bx+c using Trial and Error:** See Example 7.33.

  Step 1. Write the trinomial in descending order of degrees.
  Step 2. Find all the factor pairs of the first term.
  Step 3. Find all the factor pairs of the third term.
  Step 4. Test all the possible combinations of the factors until the correct product is found.
  Step 5. Check by multiplying.

- **Factor Trinomials of the Form ax^2+bx+c Using the "ac" Method:** See Example 7.38.

  Step 1. Factor any GCF.
  Step 2. Find the product ac.
  Step 3. Find two numbers *m* and *n* that:

Multiply to ac	$m \cdot n = a \cdot c$
Add to b	$m + n = b$

  Step 4. Split the middle term using *m* and *n*:

  $$ax^2 + bx + c$$
  $$ax^2 + \overbrace{mx + nx}^{bx} + c$$

  Step 5. Factor by grouping.
  Step 6. Check by multiplying the factors.

- **Choose a strategy to factor polynomials completely (updated):**

  Step 1. Is there a greatest common factor? Factor it.
  Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?
  If it is a binomial, right now we have no method to factor it.
  If it is a trinomial of the form x^2+bx+c
  Undo FOIL $(x\quad)(x\quad)$.
  If it is a trinomial of the form ax^2+bx+c
  Use Trial and Error or the "ac" method.
  If it has more than three terms
  Use the grouping method.
  Step 3. Check by multiplying the factors.

7.4 Factor Special Products

- **Factor perfect square trinomials** See Example 7.42.

Step 1. Does the trinomial fit he pattern?	$a^2+2ab+b^2$	$a^2-2ab+b^2$
Is the fir t term a perfect square? Write it as a square.	$(a)^2$	$(a)^2$
Is the last term a perfect square? Write it as a square.	$(a)^2 \qquad (b)^2$	$(a)^2 \qquad (b)^2$
Check the middle term. Is it $2ab$?	$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$	$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$
Step 2. Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3. Check by multiplying.		

- **Factor differences of squares** See Example 7.47.

Step 1. Does the binomial fit he pattern? Is this a diffe ence? Are the fir t and last terms perfect squares?	$a^2 - b^2$ $\_\_\_\_ - \_\_\_\_$
Step 2. Write them as squares.	$(a)^2 - (b)^2$
Step 3. Write the product of conjugates.	$(a - b)(a + b)$
Step 4. Check by multiplying.	

- **Factor sum and difference of cubes** To factor the sum or difference of cubes: See **Example 7.54**.

  Step 1. Does the binomial fit the sum or difference of cubes pattern? Is it a sum or difference? Are the first and last terms perfect cubes?

  Step 2. Write them as cubes.

  Step 3. Use either the sum or difference of cubes pattern.

  Step 4. Simplify inside the parentheses

  Step 5. Check by multiplying the factors.

7.5 General Strategy for Factoring Polynomials

- **General Strategy for Factoring Polynomials See** **Figure 7.4**.
- **How to Factor Polynomials**

  Step 1. Is there a greatest common factor? Factor it out.

  Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?

  - If it is a binomial:
    Is it a sum?
    - Of squares? Sums of squares do not factor.
    - Of cubes? Use the sum of cubes pattern.

    Is it a difference?
    - Of squares? Factor as the product of conjugates.
    - Of cubes? Use the difference of cubes pattern.
  - If it is a trinomial:
    Is it of the form $x^2 + bx + c$? Undo FOIL.
    Is it of the form $ax^2 + bx + c$?
    - If 'a' and 'c' are squares, check if it fits the trinomial square pattern.
    - Use the trial and error or 'ac' method.
  - If it has more than three terms:
    Use the grouping method.

  Step 3. Check. Is it factored completely? Do the factors multiply back to the original polynomial?

7.6 Quadratic Equations

- **Zero Product Property** If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both. See **Example 7.69**.
- **Solve a quadratic equation by factoring** To solve a quadratic equation by factoring: See **Example 7.73**.

  Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.

  Step 2. Factor the quadratic expression.

  Step 3. Use the Zero Product Property.

  Step 4. Solve the linear equations.

  Step 5. Check.
- **Use a problem solving strategy to solve word problems** See **Example 7.80**.

  Step 1. **Read** the problem. Make sure all the words and ideas are understood.

  Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

REVIEW EXERCISES

7.1 7.1 Greatest Common Factor and Factor by Grouping

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

363. $42, 60$

364. $450, 420$

365. $90, 150, 105$

366. $60, 294, 630$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

367. $24x - 42$

368. $35y + 84$

369. $15m^4 + 6m^2 n$

370. $24pt^4 + 16t^7$

Factor by Grouping

In the following exercises, factor by grouping.

371. $ax - ay + bx - by$

372. $x^2 y - xy^2 + 2x - 2y$

373. $x^2 + 7x - 3x - 21$

374. $4x^2 - 16x + 3x - 12$

375. $m^3 + m^2 + m + 1$

376. $5x - 5y - y + x$

7.2 7.2 Factor Trinomials of the form $x^2 + bx + c$

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

377. $u^2 + 17u + 72$

378. $a^2 + 14a + 33$

379. $k^2 - 16k + 60$

380. $r^2 - 11r + 28$

381. $y^2 + 6y - 7$

382. $m^2 + 3m - 54$

383. $s^2 - 2s - 8$

384. $x^2 - 3x - 10$

Factor Trinomials of the Form $x^2 + bxy + cy^2$

In the following examples, factor each trinomial of the form $x^2 + bxy + cy^2$.

385. $x^2 + 12xy + 35y^2$

386. $u^2 + 14uv + 48v^2$

387. $a^2 + 4ab - 21b^2$

388. $p^2 - 5pq - 36q^2$

7.3 7.3 Factoring Trinomials of the form $ax^2 + bx + c$

Recognize a Preliminary Strategy to Factor Polynomials Completely

In the following exercises, identify the best method to use to factor each polynomial.

389. $y^2 - 17y + 42$

390. $12r^2 + 32r + 5$

391. $8a^3 + 72a$

392. $4m - mn - 3n + 12$

Factor Trinomials of the Form $ax^2 + bx + c$ with a GCF

In the following exercises, factor completely.

393. $6x^2 + 42x + 60$

394. $8a^2 + 32a + 24$

395. $3n^4 - 12n^3 - 96n^2$

396. $5y^4 + 25y^2 - 70y$

Factor Trinomials Using the "ac" Method

In the following exercises, factor.

397. $2x^2 + 9x + 4$

398. $3y^2 + 17y + 10$

399. $18a^2 - 9a + 1$

400. $8u^2 - 14u + 3$

401. $15p^2 + 2p - 8$

402. $15x^2 + 6x - 2$

403. $40s^2 - s - 6$

404. $20n^2 - 7n - 3$

Factor Trinomials with a GCF Using the "ac" Method

In the following exercises, factor.

405. $3x^2 + 3x - 36$

406. $4x^2 + 4x - 8$

407. $60y^2 - 85y - 25$

408. $18a^2 - 57a - 21$

7.4 7.4 Factoring Special Products

Factor Perfect Square Trinomials

In the following exercises, factor.

409. $25x^2 + 30x + 9$

410. $16y^2 + 72y + 81$

411. $36a^2 - 84ab + 49b^2$

412. $64r^2 - 176rs + 121s^2$

413. $40x^2 + 360x + 810$

414. $75u^2 + 180u + 108$

415. $2y^3 - 16y^2 + 32y$

416. $5k^3 - 70k^2 + 245k$

Factor Differences of Squares

In the following exercises, factor.

417. $81r^2 - 25$

418. $49a^2 - 144$

419. $169m^2 - n^2$

420. $64x^2 - y^2$

421. $25p^2 - 1$

422. $1 - 16s^2$

423. $9 - 121y^2$

424. $100k^2 - 81$

425. $20x^2 - 125$

426. $18y^2 - 98$

427. $49u^3 - 9u$

428. $169n^3 - n$

Factor Sums and Differences of Cubes

In the following exercises, factor.

429. $a^3 - 125$

430. $b^3 - 216$

431. $2m^3 + 54$

432. $81x^3 + 3$

7.5 7.5 General Strategy for Factoring Polynomials

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

433. $24x^3 + 44x^2$

434. $24a^4 - 9a^3$

435. $16n^2 - 56mn + 49m^2$

436. $6a^2 - 25a - 9$

437. $5r^2 + 22r - 48$

438. $5u^4 - 45u^2$

439. $n^4 - 81$

440. $64j^2 + 225$

441. $5x^2 + 5x - 60$

442. $b^3 - 64$

443. $m^3 + 125$

444. $2b^2 - 2bc + 5cb - 5c^2$

7.6 7.6 Quadratic Equations

Use the Zero Product Property

In the following exercises, solve.

445. $(a - 3)(a + 7) = 0$

446. $(b - 3)(b + 10) = 0$

447. $3m(2m - 5)(m + 6) = 0$

448. $7n(3n + 8)(n - 5) = 0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

449. $x^2 + 9x + 20 = 0$

450. $y^2 - y - 72 = 0$

451. $2p^2 - 11p = 40$

452. $q^3 + 3q^2 + 2q = 0$

453. $144m^2 - 25 = 0$

454. $4n^2 = 36$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

455. The product of two consecutive numbers is 462. Find the numbers.

456. The area of a rectangular shaped patio 400 square feet. The length of the patio is 9 feet more than its width. Find the length and width.

PRACTICE TEST

In the following exercises, find the Greatest Common Factor in each expression.

457. $14y - 42$

458. $-6x^2 - 30x$

459. $80a^2 + 120a^3$

460. $5m(m - 1) + 3(m - 1)$

In the following exercises, factor completely.

461. $x^2 + 13x + 36$

462. $p^2 + pq - 12q^2$

463. $3a^3 - 6a^2 - 72a$

464. $s^2 - 25s + 84$

465. $5n^2 + 30n + 45$

466. $64y^2 - 49$

467. $xy - 8y + 7x - 56$

468. $40r^2 + 810$

469. $9s^2 - 12s + 4$

470. $n^2 + 12n + 36$

471. $100 - a^2$

472. $6x^2 - 11x - 10$

473. $3x^2 - 75y^2$

474. $c^3 - 1000d^3$

475. $ab - 3b - 2a + 6$

476. $6u^2 + 3u - 18$

477. $8m^2 + 22m + 5$

In the following exercises, solve.

478. $x^2 + 9x + 20 = 0$

479. $y^2 = y + 132$

480. $5a^2 + 26a = 24$

481. $9b^2 - 9 = 0$

482. $16 - m^2 = 0$

483. $4n^2 + 19 + 21 = 0$

484. $(x - 3)(x + 2) = 6$

485. The product of two consecutive integers is 156. Find the integers.

486. The area of a rectangular place mat is 168 square inches. Its length is two inches longer than the width. Find the length and width of the placemat.

8 RATIONAL EXPRESSIONS AND EQUATIONS

Figure 8.1 Rowing a boat downstream can be very relaxing, but it takes much more effort to row the boat upstream.

Chapter Outline

Introduction

Like rowing a boat, riding a bicycle is a situation in which going in one direction, downhill, is easy, but going in the opposite direction, uphill, can be more work. The trip to reach a destination may be quick, but the return trip whether upstream or uphill will take longer.

Rational equations are used to model situations like these. In this chapter, we will work with rational expressions, solve rational equations, and use them to solve problems in a variety of applications.

8.1 Simplify Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Evaluate rational expressions
- Simplify rational expressions
- Simplify rational expressions with opposite factors

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Simplify: $\frac{90y}{15y^2}$.

If you missed this problem, review **Example 6.66**.

2. Factor: $6x^2 - 7x + 2$.
If you missed this problem, review **Example 7.34**.

3. Factor: $n^3 + 8$.
If you missed this problem, review **Example 7.54**.

In Chapter 1, we reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers, and the denominator is not zero.

In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call these rational expressions.

Rational Expression

A **rational expression** is an expression of the form $\frac{p(x)}{q(x)}$, where *p* and *q* are polynomials and $q \neq 0$.

Remember, division by 0 is undefined.

Here are some examples of rational expressions:

$$-\frac{13}{42} \qquad \frac{7y}{8z} \qquad \frac{5x+2}{x^2-7} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{13}{42}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will perform same operations with rational expressions that we do with fractions. We will simplify, add, subtract, multiply, divide, and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

When we work with a numerical fraction, it is easy to avoid dividing by zero, because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

HOW TO : : DETERMINE THE VALUES FOR WHICH A RATIONAL EXPRESSION IS UNDEFINED.

Step 1. Set the denominator equal to zero.

Step 2. Solve the equation in the set of reals, if possible.

EXAMPLE 8.1

Determine the values for which the rational expression is undefined:

ⓐ $\frac{9y}{x}$ ⓑ $\frac{4b-3}{2b+5}$ ⓒ $\frac{x+4}{x^2+5x+6}$

Solution

The expression will be undefined when the denominator is zero.

ⓐ

	$\frac{9y}{x}$
Set the denominator equal to zero. Solve for the variable.	$x = 0$
	$\frac{9y}{x}$ is undefined or $x = 0$.

ⓑ

	$\frac{4b-3}{2b+5}$
Set the denominator equal to zero. Solve for the variable.	$\begin{aligned} 2b+5 &= 0 \\ 2b &= -5 \\ b &= -\frac{5}{2} \end{aligned}$
	$\frac{4b-3}{2b+5}$ is undefined or $b = -\frac{5}{2}$.

ⓒ

	$\frac{x+4}{x^2+5x+6}$
Set the denominator equal to zero. Solve for the variable.	$\begin{aligned} x^2+5x+6 &= 0 \\ (x+2)(x+3) &= 0 \\ x+2=0 \text{ or } x+3 &= 0 \\ x=-2 \text{ or } x &= -3 \end{aligned}$
	$\frac{x+4}{x^2+5x+6}$ is undefined or $x = -2$ or $x = -3$.

Saying that the rational expression $\frac{x+4}{x^2+5x+6}$ is undefined for $x = -2$ or $x = -3$ is similar to writing the phrase "void where prohibited" in contest rules.

TRY IT : : 8.1 Determine the values for which the rational expression is undefined:

ⓐ $\frac{3y}{x}$ ⓑ $\frac{8n-5}{3n+1}$ ⓒ $\frac{a+10}{a^2+4a+3}$

TRY IT : : 8.2 Determine the values for which the rational expression is undefined:

ⓐ $\frac{4p}{5q}$ ⓑ $\frac{y-1}{3y+2}$ ⓒ $\frac{m-5}{m^2+m-6}$

Evaluate Rational Expressions

To evaluate a rational expression, we substitute values of the variables into the expression and simplify, just as we have for many other expressions in this book.

EXAMPLE 8.2

Evaluate $\frac{2x+3}{3x-5}$ for each value:

ⓐ $x = 0$ ⓑ $x = 2$ ⓒ $x = -3$

Solution

ⓐ

	$\frac{2x+3}{3x-5}$
Substitute 0 for x.	$\frac{2(0)+3}{3(0)-5}$
Simplify.	$-\frac{3}{5}$

ⓑ

	$\frac{2x+3}{3x-5}$
Substitute 2 for x.	$\frac{2(2)+3}{3(2)-5}$
Simplify.	$\frac{4+3}{6-5}$
	$\frac{7}{1}$
	7

ⓒ

	$\frac{2x+3}{3x-5}$
Substitute -3 for x.	$\frac{2(-3)+3}{3(-3)-5}$
Simplify.	$\frac{-6+3}{-9-5}$
	$\frac{-3}{-14}$
	$\frac{3}{14}$

TRY IT : : 8.3

Evaluate $\frac{y+1}{2y-3}$ for each value:

ⓐ $y = 1$ ⓑ $y = -3$ ⓒ $y = 0$

TRY IT : : 8.4

Evaluate $\frac{5x-1}{2x+1}$ for each value:

ⓐ $x = 1$ ⓑ $x = -1$ ⓒ $x = 0$

EXAMPLE 8.3

Evaluate $\frac{x^2+8x+7}{x^2-4}$ for each value:

ⓐ $x = 0$ ⓑ $x = 2$ ⓒ $x = -1$

Solution

ⓐ

	$\frac{x^2+8x+7}{x^2-4}$
Substitute 0 for x.	$\frac{(0)^2+8(0)+7}{(0)^2-4}$
Simplify.	$\frac{7}{-4}$
	$-\frac{7}{4}$

ⓑ

	$\frac{x^2+8x+7}{x^2-4}$
Substitute 2 for x.	$\frac{(2)^2+8(2)+7}{(2)^2-4}$
Simplify.	$\frac{4+16+7}{4-4}$
	$\frac{27}{0}$
This rational expression is undefined for $x = 2$.	

ⓒ

	$\frac{x^2+8x+7}{x^2-4}$
Substitute -1 for x.	$\frac{(-1)^2+8(-1)+7}{(-1)^2-4}$
Simplify.	$\frac{1-8+7}{1-4}$
	$\frac{-7+7}{-3}$
	$\frac{0}{-3}$
	0

TRY IT : : 8.5

Evaluate $\frac{x^2+1}{x^2-3x+2}$ for each value:

ⓐ $x=0$ ⓑ $x=-1$ ⓒ $x=3$

TRY IT : : 8.6

Evaluate $\frac{x^2+x-6}{x^2-9}$ for each value:

ⓐ $x=0$ ⓑ $x=-2$ ⓒ $x=1$

Remember that a fraction is simplified when it has no common factors, other than 1, in its numerator and denominator. When we evaluate a rational expression, we make sure to simplify the resulting fraction.

EXAMPLE 8.4

Evaluate $\frac{a^2 + 2ab + b^2}{3ab^3}$ for each value:

ⓐ $a = 1,\ b = 2$ ⓑ $a = -2,\ b = -1$ ⓒ $a = \frac{1}{3},\ b = 0$

Solution

ⓐ

	$\frac{a^2 + 2ab + b^2}{3ab^2}$ when $a = 1,\ b = 2$.
Substitute 1 for *a* and 2 for *b*.	$\frac{(1)^2 + 2(1)(2) + (2)^2}{3(1)(2)^2}$
Simplify.	$\frac{1 + 4 + 4}{3(4)}$
	$\frac{9}{12}$
	$\frac{3}{4}$

ⓑ

	$\frac{a^2 + 2ab + b^2}{3ab^2}$ when $a = -2,\ b = -1$.
Substitute -2 for *a* and -1 for *b*.	$\frac{(-2)^2 + 2(-2)(-1) + (-1)^2}{3(-2)(-1)^2}$
Simplify.	$\frac{4 + 4 + 1}{-6}$
	$-\frac{9}{6}$
	$-\frac{3}{2}$

ⓒ

	$\frac{a^2 + 2ab + b^2}{3ab^2}$ when $a = \frac{1}{3},\ b = 0$.
Substitute $\frac{1}{3}$ for *a* and 0 for *b*.	$\frac{\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)(0) + (0)^2}{3\left(\frac{1}{3}\right)(0)^2}$

Simplify.	$\dfrac{\frac{1}{9}+0+0}{0}$
	$\dfrac{\frac{1}{9}}{0}$
	The expression is undefined.

TRY IT : : 8.7

Evaluate $\dfrac{2a^3b}{a^2+2ab+b^2}$ for each value:

ⓐ $a=-1,\ b=2$ ⓑ $a=0,\ b=-1$ ⓒ $a=1,\ b=\frac{1}{2}$

TRY IT : : 8.8

Evaluate $\dfrac{a^2-b^2}{8ab^3}$ for each value:

ⓐ $a=1,\ b=-1$ ⓑ $a=\frac{1}{2},\ b=-1$ ⓒ $a=-2,\ b=1$

Simplify Rational Expressions

Just like a fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator, a rational expression is *simplified* if it has no common factors, other than 1, in its numerator and denominator.

Simplified Rational Expression

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example:

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{2x}{3x}$ is not simplified because *x* is a common factor of 2*x* and 3*x*.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

Equivalent Fractions Property

If *a*, *b*, and *c* are numbers where $b \neq 0,\ c \neq 0$, then $\frac{a}{b}=\frac{a\cdot c}{b\cdot c}$ and $\frac{a\cdot c}{b\cdot c}=\frac{a}{b}$.

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0,\ c \neq 0$ clearly stated. Every time we write a rational expression, we should make a similar statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

Let's start by reviewing how we simplify numerical fractions.

EXAMPLE 8.5

Simplify: $-\frac{36}{63}$.

Solution

	$-\frac{36}{63}$
Rewrite the numerator and denominator showing the common factors.	$-\frac{4 \cdot 9}{7 \cdot 9}$
Simplify using the Equivalent Fractions Property.	$-\frac{4}{7}$

Notice that the fraction $-\frac{4}{7}$ is simplified because there are no more common factors.

TRY IT : : 8.9 Simplify: $-\frac{45}{81}$.

TRY IT : : 8.10 Simplify: $-\frac{42}{54}$.

Throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$ and $y \neq 0$.

EXAMPLE 8.6

Simplify: $\frac{3xy}{18x^2y^2}$.

Solution

	$\frac{3xy}{18x^2y^2}$
Rewrite the numerator and denominator showing the common factors.	$\frac{1 \cdot 3xy}{6xy \cdot 3xy}$
Simplify using the Equivalent Fractions Property.	$\frac{1}{6xy}$

Did you notice that these are the same steps we took when we divided monomials in **Polynomials**?

TRY IT : : 8.11 Simplify: $\frac{4x^2y}{12xy^2}$.

TRY IT : : 8.12 Simplify: $\frac{16x^2y}{2xy^2}$.

To simplify rational expressions we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\dfrac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$	$\dfrac{3x\cancel{(x-9)}}{5\cancel{(x-9)}}$	$\dfrac{x+5}{x}$
$\dfrac{2}{5}$	$\dfrac{3x}{5}$	**NO COMMON FACTORS**
We removed the common factors 3 and 7. They are factors of the product.	We removed the common factor $(x-9)$. It is a factor of the product.	While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Note that removing the x's from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$!

EXAMPLE 8.7 HOW TO SIMPLIFY RATIONAL BINOMIALS

Simplify: $\dfrac{2x+8}{5x+20}$.

Solution

Step 1. Factor the numerator and denominator completely.	Factor $2x+8$ and $5x-20$.	$\dfrac{2x+8}{5x+20}$ $\dfrac{2(x+4)}{5(x+4)}$
Step 2. Simplify by dividing out common factors.	Divide out the common factors.	$\dfrac{2\cancel{(x+4)}}{5\cancel{(x+4)}}$ $\dfrac{2}{5}$

TRY IT : : 8.13 Simplify: $\dfrac{3x-6}{2x-4}$.

TRY IT : : 8.14 Simplify: $\dfrac{7y+35}{5y+25}$.

We now summarize the steps you should follow to simplify rational expressions.

HOW TO : : SIMPLIFY A RATIONAL EXPRESSION.

Step 1. Factor the numerator and denominator completely.

Step 2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way it is easy to check that we have removed all the common factors!

We'll use the methods we covered in **Factoring** to factor the polynomials in the numerators and denominators in the following examples.

EXAMPLE 8.8

Simplify: $\dfrac{x^2+5x+6}{x^2+8x+12}$.

Solution

	$\dfrac{x^2+5x+6}{x^2+8x+12}$
Factor the numerator and denominator.	$\dfrac{(x+2)(x+3)}{(x+2)(x+6)}$
Remove the common factor $x+2$ from the numerator and the denominator.	$\dfrac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$
	$\dfrac{x+3}{x+6}$

Can you tell which values of *x* must be excluded in this example?

TRY IT : : 8.15 Simplify: $\dfrac{x^2-x-2}{x^2-3x+2}$.

TRY IT : : 8.16 Simplify: $\dfrac{x^2-3x-10}{x^2+x-2}$.

EXAMPLE 8.9

Simplify: $\dfrac{y^2+y-42}{y^2-36}$.

Solution

	$\dfrac{y^2+y-42}{y^2-36}$
Factor the numerator and denominator.	$\dfrac{(y+7)(y-6)}{(y+6)(y-6)}$
Remove the common factor $y-6$ from the numerator and the denominator.	$\dfrac{(y+7)\cancel{(y-6)}}{(y+6)\cancel{(y-6)}}$
	$\dfrac{y+7}{y+6}$

TRY IT : : 8.17 Simplify: $\dfrac{x^2+x-6}{x^2-4}$.

TRY IT : : 8.18 Simplify: $\dfrac{x^2+8x+7}{x^2-49}$.

EXAMPLE 8.10

Simplify: $\dfrac{p^3-2p^2+2p-4}{p^2-7p+10}$.

Solution

$$\frac{p^3 - 2p^2 + 2p - 4}{p^2 - 7p + 10}$$

Factor the numerator and denominator, using grouping to factor the numerator.

$$\frac{p^2(p-2) + 2(p-2)}{(p-5)(p-2)}$$

$$\frac{(p^2+2)(p-2)}{(p-5)(p-2)}$$

Remove the common factor of $p - 2$ from the numerator and the denominator.

$$\frac{(p^2+2)\cancel{(p-2)}}{(p-5)\cancel{(p-2)}}$$

$$\frac{p^2+2}{p-5}$$

TRY IT : : 8.19 Simplify: $\dfrac{y^3 - 3y^2 + y - 3}{y^2 - y - 6}$.

TRY IT : : 8.20 Simplify: $\dfrac{p^3 - p^2 + 2p - 2}{p^2 + 4p - 5}$.

EXAMPLE 8.11

Simplify: $\dfrac{2n^2 - 14n}{4n^2 - 16n - 48}$.

Solution

$$\frac{2n^2 - 14n}{4n^2 - 16n - 48}$$

Factor the numerator and denominator, fir t factoring out the GCF.

$$\frac{2n(n-7)}{4\left(n^2 - 4n - 12\right)}$$

$$\frac{2n(n-7)}{4(n-6)(n+2)}$$

Remove the common factor, 2.

$$\frac{\cancel{2}n(n-7)}{\cancel{2}\cdot 2(n-6)(n+2)}$$

$$\frac{n(n-7)}{2(n-6)(n+2)}$$

TRY IT : : 8.21 Simplify: $\dfrac{2n^2 - 10n}{4n^2 - 16n - 20}$.

TRY IT : : 8.22 Simplify: $\dfrac{4x^2 - 16x}{8x^2 - 16x - 64}$.

EXAMPLE 8.12

Simplify: $\dfrac{3b^2 - 12b + 12}{6b^2 - 24}$.

Solution

	$\dfrac{3b^2 - 12b + 12}{6b^2 - 24}$
Factor the numerator and denominator, fir t factoring out the GCF.	$\dfrac{3\left(b^2 - 4b + 4\right)}{6\left(b^2 - 4\right)}$
	$\dfrac{3(b-2)(b-2)}{6(b+2)(b-2)}$
Remove the common factors of $b - 2$ and 3.	$\dfrac{\cancel{3}(b-2)\cancel{(b-2)}}{\cancel{3}\cdot 2(b+2)\cancel{(b-2)}}$
	$\dfrac{b-2}{2(b+2)}$

TRY IT : : 8.23 Simplify: $\dfrac{2x^2 - 12x + 18}{3x^2 - 27}$.

TRY IT : : 8.24 Simplify: $\dfrac{5y^2 - 30y + 25}{2y^2 - 50}$.

EXAMPLE 8.13

Simplify: $\dfrac{m^3 + 8}{m^2 - 4}$.

Solution

	$\dfrac{m^3 + 8}{m^2 - 4}$
Factor the numerator and denominator, using the formulas for sum of cubes and diffe ence of squares.	$\dfrac{(m+2)\left(m^2 - 2m + 4\right)}{(m+2)(m-2)}$
Remove the common factor of $m + 2$.	$\dfrac{\cancel{(m+2)}\left(m^2 - 2m + 4\right)}{\cancel{(m+2)}(m-2)}$
	$\dfrac{m^2 - 2m + 4}{m - 2}$

TRY IT : : 8.25 Simplify: $\dfrac{p^3 - 64}{p^2 - 16}$.

TRY IT : : 8.26 Simplify: $\dfrac{x^3 + 8}{x^2 - 4}$.

Simplify Rational Expressions with Opposite Factors

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. Let’s start with a numerical fraction, say $\frac{7}{-7}$. We know this fraction simplifies to -1. We also recognize that the numerator and denominator are opposites.

In **Foundations**, we introduced opposite notation: the opposite of a is $-a$. We remember, too, that $-a = -1 \cdot a$.

We simplify the fraction $\frac{a}{-a}$, whose numerator and denominator are opposites, in this way:

	$\frac{a}{-a}$
We could rewrite this.	$\frac{1 \cdot a}{-1 \cdot a}$
Remove the common factors.	$\frac{1}{-1}$
Simplify.	-1

So, in the same way, we can simplify the fraction $\frac{x-3}{-(x-3)}$:

We could rewrite this.	$\frac{1 \cdot (x-3)}{-1 \cdot (x-3)}$
Remove the common factors.	$\frac{1}{-1}$
Simplify.	-1

But the opposite of $x-3$ could be written differently:

	$-(x-3)$
Distribute.	$-x+3$
Rewrite.	$3-x$

This means the fraction $\frac{x-3}{3-x}$ simplifies to -1.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $\frac{a-b}{b-a}$ simplifies to -1.

Opposites in a Rational Expression

The opposite of $a-b$ is $b-a$.

$$\frac{a-b}{b-a} = -1 \qquad a \neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators.

EXAMPLE 8.14

Simplify: $\frac{x-8}{8-x}$.

Solution

	$\frac{x-8}{8-x}$
Recognize that $x-8$ and $8-x$ are opposites.	-1

TRY IT : : 8.27 Simplify: $\frac{y-2}{2-y}$.

TRY IT : : 8.28 Simplify: $\frac{n-9}{9-n}$.

Remember, the first step in simplifying a rational expression is to factor the numerator and denominator completely.

EXAMPLE 8.15

Simplify: $\frac{14-2x}{x^2-49}$.

Solution

	$\frac{14-2x}{x^2-49}$
Factor the numerator and denominator.	$\frac{2(7-x)}{(x+7)(x-7)}$
Recognize that $7-x$ and $x-7$ are opposites .	$\frac{2\cancel{(7-x)}}{(x+7)\cancel{(x-7)}}(-1)$
Simplify.	$-\frac{2}{x+7}$

TRY IT : : 8.29 Simplify: $\frac{10-2y}{y^2-25}$.

TRY IT : : 8.30 Simplify: $\frac{3y-27}{81-y^2}$.

EXAMPLE 8.16

Simplify: $\frac{x^2-4x-32}{64-x^2}$.

Solution

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$(-1)\frac{\cancel{(x-8)}(x+4)}{\cancel{(8-x)}(8+x)}$
Simplify.	$-\frac{x+4}{x+8}$

TRY IT : : 8.31 Simplify: $\frac{x^2-4x-5}{25-x^2}$.

TRY IT : : 8.32 Simplify: $\frac{x^2+x-2}{1-x^2}$.

8.1 EXERCISES

Practice Makes Perfect

In the following exercises, determine the values for which the rational expression is undefined.

1.

ⓐ $\frac{2x}{z}$

ⓑ $\frac{4p-1}{6p-5}$

ⓒ $\frac{n-3}{n^2+2n-8}$

2.

ⓐ $\frac{10m}{11n}$

ⓑ $\frac{6y+13}{4y-9}$

ⓒ $\frac{b-8}{b^2-36}$

3.

ⓐ $\frac{4x^2y}{3y}$

ⓑ $\frac{3x-2}{2x+1}$

ⓒ $\frac{u-1}{u^2-3u-28}$

4.

ⓐ $\frac{5pq^2}{9q}$

ⓑ $\frac{7a-4}{3a+5}$

ⓒ $\frac{1}{x^2-4}$

Evaluate Rational Expressions

In the following exercises, evaluate the rational expression for the given values.

5. $\frac{2x}{x-1}$

ⓐ $x=0$

ⓑ $x=2$

ⓒ $x=-1$

6. $\frac{4y-1}{5y-3}$

ⓐ $y=0$

ⓑ $y=2$

ⓒ $y=-1$

7. $\frac{2p+3}{p^2+1}$

ⓐ $p=0$

ⓑ $p=1$

ⓒ $p=-2$

8. $\frac{x+3}{2-3x}$

ⓐ $x=0$

ⓑ $x=1$

ⓒ $x=-2$

9. $\frac{y^2+5y+6}{y^2-1}$

ⓐ $y=0$

ⓑ $y=2$

ⓒ $y=-2$

10. $\frac{z^2+3z-10}{z^2-1}$

ⓐ $z=0$

ⓑ $z=2$

ⓒ $z=-2$

11. $\frac{a^2-4}{a^2+5a+4}$

ⓐ $a=0$

ⓑ $a=1$

ⓒ $a=-2$

12. $\frac{b^2+2}{b^2-3b-4}$

ⓐ $b=0$

ⓑ $b=2$

ⓒ $b=-2$

13. $\frac{x^2+3xy+2y^2}{2x^3y}$

ⓐ $x=1,\ y=-1$

ⓑ $x=2,\ y=1$

ⓒ $x=-1,\ y=-2$

14. $\dfrac{c^2 + cd - 2d^2}{cd^3}$

ⓐ $c = 2,\ d = -1$

ⓑ $c = 1,\ d = -1$

ⓒ $c = -1,\ d = 2$

15. $\dfrac{m^2 - 4n^2}{5mn^3}$

ⓐ $m = 2,\ n = 1$

ⓑ $m = -1,\ n = -1$

ⓒ $m = 3,\ n = 2$

16. $\dfrac{2s^2 t}{s^2 - 9t^2}$

ⓐ $s = 4,\ t = 1$

ⓑ $s = -1,\ t = -1$

ⓒ $s = 0,\ t = 2$

Simplify Rational Expressions

In the following exercises, simplify.

17. $-\dfrac{4}{52}$

18. $-\dfrac{44}{55}$

19. $\dfrac{56}{63}$

20. $\dfrac{65}{104}$

21. $\dfrac{6ab^2}{12a^2 b}$

22. $\dfrac{15xy}{3x^3 y^3}$

23. $\dfrac{8m^3 n}{12mn^2}$

24. $\dfrac{36v^3 w^2}{27vw^3}$

25. $\dfrac{3a + 6}{4a + 8}$

26. $\dfrac{5b + 5}{6b + 6}$

27. $\dfrac{3c - 9}{5c - 15}$

28. $\dfrac{4d + 8}{9d + 18}$

29. $\dfrac{7m + 63}{5m + 45}$

30. $\dfrac{8n - 96}{3n - 36}$

31. $\dfrac{12p - 240}{5p - 100}$

32. $\dfrac{6q + 210}{5q + 175}$

33. $\dfrac{a^2 - a - 12}{a^2 - 8a + 16}$

34. $\dfrac{x^2 + 4x - 5}{x^2 - 2x + 1}$

35. $\dfrac{y^2 + 3y - 4}{y^2 - 6y + 5}$

36. $\dfrac{v^2 + 8v + 15}{v^2 - v - 12}$

37. $\dfrac{x^2 - 25}{x^2 + 2x - 15}$

38. $\dfrac{a^2 - 4}{a^2 + 6a - 16}$

39. $\dfrac{y^2 - 2y - 3}{y^2 - 9}$

40. $\dfrac{b^2 + 9b + 18}{b^2 - 36}$

41. $\dfrac{y^3 + y^2 + y + 1}{y^2 + 2y + 1}$

42. $\dfrac{p^3 + 3p^2 + 4p + 12}{p^2 + p - 6}$

43. $\dfrac{x^3 - 2x^2 - 25x + 50}{x^2 - 25}$

44. $\dfrac{q^3 + 3q^2 - 4q - 12}{q^2 - 4}$

45. $\dfrac{3a^2 + 15a}{6a^2 + 6a - 36}$

46. $\dfrac{8b^2 - 32b}{2b^2 - 6b - 80}$

47. $\dfrac{-5c^2 - 10c}{-10c^2 + 30c + 100}$

48. $\dfrac{4d^2 - 24d}{2d^2 - 4d - 48}$

49. $\dfrac{3m^2 + 30m + 75}{4m^2 - 100}$

50. $\dfrac{5n^2 + 30n + 45}{2n^2 - 18}$

51. $\dfrac{5r^2 + 30r - 35}{r^2 - 49}$

52. $\dfrac{3s^2 + 30s + 24}{3s^2 - 48}$

53. $\dfrac{t^3 - 27}{t^2 - 9}$

54. $\dfrac{v^3 - 1}{v^2 - 1}$

55. $\dfrac{w^3 + 216}{w^2 - 36}$

56. $\dfrac{v^3 + 125}{v^2 - 25}$

Simplify Rational Expressions with Opposite Factors

In the following exercises, simplify each rational expression.

57. $\dfrac{a-5}{5-a}$

58. $\dfrac{b-12}{12-b}$

59. $\dfrac{11-c}{c-11}$

60. $\dfrac{5-d}{d-5}$

61. $\dfrac{12-2x}{x^2-36}$

62. $\dfrac{20-5y}{y^2-16}$

63. $\dfrac{4v-32}{64-v^2}$

64. $\dfrac{7w-21}{9-w^2}$

65. $\dfrac{y^2-11y+24}{9-y^2}$

66. $\dfrac{z^2-9z+20}{16-z^2}$

67. $\dfrac{a^2-5z-36}{81-a^2}$

68. $\dfrac{b^2+b-42}{36-b^2}$

Everyday Math

69. Tax Rates For the tax year 2015, the amount of tax owed by a single person earning between $37,450 and $90,750, can be found by evaluating the formula $0.25x - 4206.25,$ where *x* is income. The average tax rate for this income can be found by evaluating the formula $\dfrac{0.25x - 4206.25}{x}.$ What would be the average tax rate for a single person earning $50,000?

70. Work The length of time it takes for two people for perform the same task if they work together can be found by evaluating the formula $\dfrac{xy}{x+y}.$ If Tom can paint the den in $x =$ 45 minutes and his brother Bobby can paint it in $y =$ 60 minutes, how many minutes will it take them if they work together?

Writing Exercises

71. Explain how you find the values of *x* for which the rational expression $\dfrac{x^2-x-20}{x^2-4}$ is undefined.

72. Explain all the steps you take to simplify the rational expression $\dfrac{p^2+4p-21}{9-p^2}.$

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
evaluate rational expressions.			
simplify rational expressions.			
simplify rational expressions with opposite factors.			

ⓑ *If most of your checks were:*

…confidently. *Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!*

…with some help. *This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math* *tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.*

8.2 Multiply and Divide Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Multiply rational expressions
- Divide rational expressions

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$.
   If you missed this problem, review **Example 1.68**.
2. Divide: $\frac{14}{15} \div \frac{6}{35}$.
   If you missed this problem, review **Example 1.71**.
3. Factor completely: $2x^2 - 98$.
   If you missed this problem, review **Example 7.62**.
4. Factor completely: $10n^3 + 10$.
   If you missed this problem, review **Example 7.65**.
5. Factor completely: $10p^2 - 25pq - 15q^2$.
   If you missed this problem, review **Example 7.68**.

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Multiplication of Rational Expressions

If p, q, r, s are polynomials where $q \neq 0$ and $s \neq 0$, then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

We'll do the first example with numerical fractions to remind us of how we multiplied fractions without variables.

EXAMPLE 8.17

Multiply: $\frac{10}{28} \cdot \frac{8}{15}$.

Solution

	$\frac{10}{28} \cdot \frac{8}{15}$
Multiply the numerators and denominators.	$\frac{10 \cdot 8}{28 \cdot 15}$
Look for common factors, and then remove them.	$\frac{2 \cdot \cancel{5} \cdot 2 \cdot \cancel{4}}{7 \cdot \cancel{4} \cdot 3 \cdot \cancel{5}}$
Simplify.	$\frac{4}{21}$

TRY IT : : 8.33 Mulitply: $\frac{6}{10} \cdot \frac{15}{12}$.

TRY IT : : 8.34 Mulitply: $\frac{20}{15} \cdot \frac{6}{8}$.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$ and $y \neq 0$.

EXAMPLE 8.18

Mulitply: $\frac{2x}{3y^2} \cdot \frac{6xy^3}{x^2 y}$.

Solution

	$\frac{2x}{3y^2} \cdot \frac{6xy^3}{x^2 y}$
Multiply.	$\frac{2x \cdot 6xy^3}{3y^2 \cdot x^2 y}$
Factor the numerator and denominator completely, and then remove common factors.	$\frac{2 \cdot \cancel{x} \cdot 2 \cdot \cancel{3} \cdot \cancel{x} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y}}{\cancel{3} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{y}}$
Simplify.	4

TRY IT : : 8.35 Mulitply: $\frac{3pq}{q^2} \cdot \frac{5p^2 q}{6pq}$.

TRY IT : : 8.36 Mulitply: $\frac{6x^3 y}{7x^2} \cdot \frac{2xy^3}{x^2 y}$.

EXAMPLE 8.19 HOW TO MULTIPLY RATIONAL EXPRESSIONS

Mulitply: $\frac{2x}{x^2 + x + 12} \cdot \frac{x^2 - 9}{6x^2}$.

Solution

Step 1. Factor the numerator and denominator completely.	Factor $x^2 - 9$ and $x^2 + x + 12$.	$\frac{2x}{x^2 + x + 12} \cdot \frac{x^2 - 9}{6x^2}$ $\frac{2x}{(x-3)(x-4)} \cdot \frac{(x-3)(x+3)}{6x^2}$
Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\frac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\frac{\cancel{2}\cancel{x}\,\cancel{(x-3)}(x+3)}{\cancel{2} \cdot 3 \cdot \cancel{x} \cdot x\,\cancel{(x-3)}(x-4)}$ $\frac{(x+3)}{3x(x-4)}$

TRY IT : : 8.37

Mulitply: $\dfrac{5x}{x^2+5x+6} \cdot \dfrac{x^2-4}{10x}$.

TRY IT : : 8.38

Mulitply: $\dfrac{9x^2}{x^2+11x+30} \cdot \dfrac{x^2-36}{3x^2}$.

HOW TO : : MULTIPLY A RATIONAL EXPRESSION.

Step 1. Factor each numerator and denominator completely.
Step 2. Multiply the numerators and denominators.
Step 3. Simplify by dividing out common factors.

EXAMPLE 8.20

Multiply: $\dfrac{n^2-7n}{n^2+2n+1} \cdot \dfrac{n+1}{2n}$.

Solution

	$\dfrac{n^2-7n}{n^2+2n+1} \cdot \dfrac{n+1}{2n}$
Factor each numerator and denominator.	$\dfrac{n(n-7)}{(n+1)(n+1)} \cdot \dfrac{n+1}{2n}$
Multiply the numerators and the denominators.	$\dfrac{n(n-7)(n+1)}{(n+1)(n+1)2n}$
Remove common factors.	$\dfrac{\cancel{n}(n-7)\cancel{(n+1)}}{(n+1)\cancel{(n+1)}2\cancel{n}}$
Simplify.	$\dfrac{n-7}{2(n+1)}$

TRY IT : : 8.39

Multiply: $\dfrac{x^2-25}{x^2-3x-10} \cdot \dfrac{x+2}{x}$.

TRY IT : : 8.40

Multiply: $\dfrac{x^2-4x}{x^2+5x+6} \cdot \dfrac{x+2}{x}$.

EXAMPLE 8.21

Multiply: $\dfrac{16-4x}{2x-12} \cdot \dfrac{x^2-5x-6}{x^2-16}$.

Solution

	$\frac{16-4x}{2x-12}\cdot\frac{x^2-5x-6}{x^2-16}$
Factor each numerator and denominator.	$\frac{4(4-x)}{2(x-6)}\cdot\frac{(x-6)(x+1)}{(x-4)(x+4)}$
Multiply the numerators and the denominators.	$\frac{4(4-x)(x-6)(x+1)}{2(x-6)(x-4)(x+4)}$
Remove common factors.	$(-1)\frac{\cancel{2}\cdot 2\cancel{(4-x)}\cancel{(x-6)}(x+1)}{\cancel{2}\cancel{(x-6)}\cancel{(x-4)}(x+4)}$
Simplify.	$-\frac{2(x+1)}{(x+4)}$

TRY IT : : 8.41 Multiply: $\frac{12x-6x^2}{x^2+8x}\cdot\frac{x^2+11x+24}{x^2-4}$.

TRY IT : : 8.42 Multiply: $\frac{9v-3v^2}{9v+36}\cdot\frac{v^2+7v+12}{v^2-9}$.

EXAMPLE 8.22

Multiply: $\frac{2x-6}{x^2-8x+15}\cdot\frac{x^2-25}{2x+10}$.

Solution

	$\frac{2x-6}{x^2-8x+15}\cdot\frac{x^2-25}{2x+10}$
Factor each numerator and denominator.	$\frac{2(x-3)}{(x-3)(x-5)}\cdot\frac{(x-5)(x+5)}{2(x+5)}$
Multiply the numerators and denominators.	$\frac{2(x-3)(x-5)(x+5)}{2(x-3)(x-5)(x+5)}$
Remove common factors.	$\frac{\cancel{2}\cancel{(x-3)}\cancel{(x-5)}\cancel{(x+5)}}{\cancel{2}\cancel{(x-3)}\cancel{(x-5)}\cancel{(x+5)}}$
Simplify.	1

TRY IT : : 8.43 Multiply: $\frac{3a-21}{a^2-9a+14}\cdot\frac{a^2-4}{3a+6}$.

TRY IT : : 8.44 Multiply: $\frac{b^2-b}{b^2+9b-10}\cdot\frac{b^2-100}{b^2-10b}$.

Divide Rational Expressions

To divide rational expressions we multiply the first fraction by the reciprocal of the second, just like we did for numerical fractions.

Remember, the **reciprocal** of $\frac{a}{b}$ is $\frac{b}{a}$. To find the reciprocal we simply put the numerator in the denominator and the denominator in the numerator. We "flip" the fraction.

Division of Rational Expressions

If p, q, r, s are polynomials where $q \neq 0, r \neq 0, s \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions multiply the first fraction by the reciprocal of the second.

EXAMPLE 8.23 HOW TO DIVIDE RATIONAL EXPRESSIONS

Divide: $\frac{x+9}{6-x} \div \frac{x^2-81}{x-6}$.

Solution

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\frac{x+9}{6-x} \div \frac{x^2-81}{x-6}$ $\frac{x+9}{6-x} \cdot \frac{x-6}{x^2-81}$
Step 2. Factor the numerators and denominators completely.	Factor $x^2 - 81$.	$\frac{x+9}{6-x} \cdot \frac{x-6}{(x-9)(x+9)}$
Step 3. Multiply the numerators and denominators.		$\frac{(x+9)(x-6)}{(6-x)(x-9)(x+9)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors. Remember opposites divide to –1.	$(-1)\frac{\cancel{(x+9)}\cancel{(x-6)}}{\cancel{(6-x)}(x-9)\cancel{(x+9)}}$ $-\frac{1}{(x-9)}$

TRY IT : : 8.45 Divide: $\frac{c+3}{5-c} \div \frac{c^2-9}{c-5}$.

TRY IT : : 8.46 Divide: $\frac{2-d}{d-4} \div \frac{4-d^2}{4-d}$.

HOW TO : : DIVIDE RATIONAL EXPRESSIONS.

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.

Step 2. Factor the numerators and denominators completely.

Step 3. Multiply the numerators and denominators together.

Step 4. Simplify by dividing out common factors.

EXAMPLE 8.24

Divide: $\dfrac{3n^2}{n^2-4n} \div \dfrac{9n^2-45n}{n^2-7n+10}$.

Solution

	$\dfrac{3n^2}{n^2-4n} \div \dfrac{9n^2-45n}{n^2-7n+10}$
Rewrite the division as the product of the first rational expression and the reciprocal of the second.	$\dfrac{3n^2}{n^2-4n} \cdot \dfrac{n^2-7n+10}{9n^2-45n}$
Factor the numerators and denominators and then multiply.	$\dfrac{3\cdot n\cdot n\cdot(n-5)(n-2)}{n(n-4)\cdot 3\cdot 3\cdot n\cdot(n-5)}$
Simplify by dividing out common factors.	$\dfrac{\cancel{3}\cdot\cancel{n}\cdot\cancel{n}\cancel{(n-5)}(n-2)}{\cancel{n}(n-4)\cancel{3}\cdot 3\cdot\cancel{n}\cancel{(n-5)}}$
	$\dfrac{n-2}{3(n-4)}$

TRY IT : : 8.47 Divide: $\dfrac{2m^2}{m^2-8m} \div \dfrac{8m^2+24m}{m^2+m-6}$.

TRY IT : : 8.48 Divide: $\dfrac{15n^2}{3n^2+33n} \div \dfrac{5n-5}{n^2+9n-22}$.

Remember, first rewrite the division as multiplication of the first expression by the reciprocal of the second. Then factor everything and look for common factors.

EXAMPLE 8.25

Divide: $\dfrac{2x^2+5x-12}{x^2-16} \div \dfrac{2x^2-13x+15}{x^2-8x+16}$.

Solution

	$\dfrac{2x^2+5x-12}{x^2-16} \div \dfrac{2x^2-13x+15}{x^2-8x+16}$
Rewrite the division as multiplication of the fir t expression by the reciprocal of the second.	$\dfrac{2x^2+5x-12}{x^2-16} \cdot \dfrac{x^2-8x+16}{2x^2-13x+15}$
Factor the numerators and denominators and then multiply.	$\dfrac{(2x-3)(x+4)(x-4)(x-4)}{(x-4)(x+4)(2x-3)(x-5)}$
Simplify by dividing out common factors.	$\dfrac{\cancel{(2x-3)}\cancel{(x+4)}\cancel{(x-4)}(x-4)}{\cancel{(x-4)}\cancel{(x+4)}\cancel{(2x-3)}(x-5)}$
Simplify.	$\dfrac{x-4}{x-5}$

TRY IT : : 8.49 Divide: $\dfrac{3a^2-8a-3}{a^2-25} \div \dfrac{3a^2-14a-5}{a^2+10a+25}$.

TRY IT : : 8.50 Divide: $\dfrac{4b^2 + 7b - 2}{1 - b^2} \div \dfrac{4b^2 + 15b - 4}{b^2 - 2b + 1}$.

EXAMPLE 8.26

Divide: $\dfrac{p^3 + q^3}{2p^2 + 2pq + 2q^2} \div \dfrac{p^2 - q^2}{6}$.

Solution

$$\frac{p^3 + q^3}{2p^2 + 2pq + 2q^2} \div \frac{p^2 - q^2}{6}$$

Rewrite the division as a multiplication of the fir t expression times the reciprocal of the second.

$$\frac{p^3 + q^3}{2p^2 + 2pq + 2q^2} \cdot \frac{6}{p^2 - q^2}$$

Factor the numerators and denominators and then multiply.

$$\frac{(p+q)\left(p^2 - pq + q^2\right)6}{2\left(p^2 + pq + q^2\right)(p-q)(p+q)}$$

Simplify by dividing out common factors.

$$\frac{\cancel{(p+q)}\left(p^2 - pq + q^2\right)\cancel{6}^{3}}{\cancel{2}\left(p^2 + pq + q^2\right)(p-q)\cancel{(p+q)}}$$

Simplify.

$$\frac{3\left(p^2 - pq + q^2\right)}{(p-q)\left(p^2 + pq + q^2\right)}$$

TRY IT : : 8.51 Divide: $\dfrac{x^3 - 8}{3x^2 - 6x + 12} \div \dfrac{x^2 - 4}{6}$.

TRY IT : : 8.52 Divide: $\dfrac{2z^2}{z^2 - 1} \div \dfrac{z^3 - z^2 + z}{z^3 - 1}$.

Before doing the next example, let's look at how we divide a fraction by a whole number. When we divide $\frac{3}{5} \div 4$, we first write 4 as a fraction so that we can find its reciprocal.

$$\frac{3}{5} \div 4$$
$$\frac{3}{5} \div \frac{4}{1}$$
$$\frac{3}{5} \cdot \frac{1}{4}$$

We do the same thing when we divide rational expressions.

EXAMPLE 8.27

Divide: $\dfrac{a^2 - b^2}{3ab} \div \left(a^2 + 2ab + b^2\right)$.

Solution

	$\frac{a^2 - b^2}{3ab} \div \left(a^2 + 2ab + b^2\right)$
Write the second expression as a fraction.	$\frac{a^2 - b^2}{3ab} \div \frac{a^2 + 2ab + b^2}{1}$
Rewrite the division as the fir t expression times the reciprocal of the second expression.	$\frac{a^2 - b^2}{3ab} \cdot \frac{1}{a^2 + 2ab + b^2}$
Factor the numerators and the denominators, and then multiply.	$\frac{(a - b)(a + b) \cdot 1}{3ab \cdot (a + b)(a + b)}$
Simplify by dividing out common factors.	$\frac{(a - b)\cancel{(a + b)}}{3ab \cdot \cancel{(a + b)}(a + b)}$
Simplify.	$\frac{(a - b)}{3ab(a + b)}$

TRY IT : : 8.53 Divide: $\frac{2x^2 - 14x - 16}{4} \div (x^2 + 2x + 1)$.

TRY IT : : 8.54 Divide: $\frac{y^2 - 6y + 8}{y^2 - 4y} \div (3y^2 - 12y)$.

Remember a fraction bar means division. A complex fraction is another way of writing division of two fractions.

EXAMPLE 8.28

Divide: $\dfrac{\frac{6x^2 - 7x + 2}{4x - 8}}{\frac{2x^2 - 7x + 3}{x^2 - 5x + 6}}$.

Solution

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$$

Rewrite with a division sign.	$\frac{6x^2-7x+2}{4x-8} \div \frac{2x^2-7x+3}{x^2-5x+6}$
Rewrite as product of fir t times reciprocal of second.	$\frac{6x^2-7x+2}{4x-8} \cdot \frac{x^2-5x+6}{2x^2-7x+3}$
Factor the numerators and the denominators, and then multiply.	$\frac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$
Simplify by dividing out common factors.	$\frac{\cancel{(2x-1)}(3x-2)\cancel{(x-2)}\cancel{(x-3)}}{4\cancel{(x-2)}\cancel{(2x-1)}\cancel{(x-3)}}$
Simplify.	$\frac{3x-2}{4}$

TRY IT : : 8.55

Divide: $\frac{\frac{3x^2+7x+2}{4x+24}}{\frac{3x^2-14x-5}{x^2+x-30}}$.

TRY IT : : 8.56

Divide: $\frac{\frac{y^2-36}{2y^2+11y-6}}{\frac{2y^2-2y-60}{8y-4}}$.

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then we factor and multiply.

EXAMPLE 8.29

Divide: $\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$.

Solution

	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \cdot \frac{8x+16}{2x+12}$
Factor the numerators and the denominators, and then multiply.	$\frac{3 \cdot 8(x-2)(x+3)(x-1)(x+2)}{4 \cdot 2(x-1)(x+2)(x-5)(x+6)}$
Simplify by dividing out common factors.	$\frac{3 \cdot \cancel{8}(x-2)(x+3)\cancel{(x-1)}\cancel{(x+2)}}{\cancel{4} \cdot \cancel{2}\cancel{(x-1)}\cancel{(x+2)}(x-5)(x+6)}$
Simplify.	$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

TRY IT : : 8.57 Divide: $\frac{4m+4}{3m-15} \cdot \frac{m^2-3m-10}{m^2-4m-32} \div \frac{12m-36}{6m-48}$.

TRY IT : : 8.58 Divide: $\frac{2n^2+10n}{n-1} \div \frac{n^2+10n+24}{n^2+8n-9} \cdot \frac{n+4}{8n^2+12n}$.

8.2 EXERCISES

Practice Makes Perfect

Multiply Rational Expressions

In the following exercises, multiply.

73. $\frac{12}{16} \cdot \frac{4}{10}$

74. $\frac{32}{5} \cdot \frac{16}{24}$

75. $\frac{18}{10} \cdot \frac{4}{30}$

76. $\frac{21}{36} \cdot \frac{45}{24}$

77. $\frac{5x^2y^4}{12xy^3} \cdot \frac{6x^2}{20y^2}$

78. $\frac{8w^3y}{9y^2} \cdot \frac{3y}{4w^4}$

79. $\frac{12a^3b}{b^2} \cdot \frac{2ab^2}{9b^3}$

80. $\frac{4mn^2}{5n^3} \cdot \frac{mn^3}{8m^2n^2}$

81. $\frac{5p^2}{p^2-5p-36} \cdot \frac{p^2-16}{10p}$

82. $\frac{3q^2}{q^2+q-6} \cdot \frac{q^2-9}{9q}$

83. $\frac{4r}{r^2-3r-10} \cdot \frac{r^2-25}{8r^2}$

84. $\frac{s}{s^2-9s+14} \cdot \frac{s^2-49}{7s^2}$

85. $\frac{x^2-7x}{x^2+6x+9} \cdot \frac{x+3}{4x}$

86. $\frac{2y^2-10y}{y^2+10y+25} \cdot \frac{y+5}{6y}$

87. $\frac{z^2+3z}{z^2-3z-4} \cdot \frac{z-4}{z^2}$

88. $\frac{2a^2+8a}{a^2-9a+20} \cdot \frac{a-5}{a^2}$

89. $\frac{28-4b}{3b-3} \cdot \frac{b^2+8b-9}{b^2-49}$

90. $\frac{18c-2c^2}{6c+30} \cdot \frac{c^2+7c+10}{c^2-81}$

91. $\frac{35d-7d^2}{d^2+7d} \cdot \frac{d^2+12d+35}{d^2-25}$

92. $\frac{72m-12m^2}{8m+32} \cdot \frac{m^2+10m+24}{m^2-36}$

93. $\frac{4n+20}{n^2+n-20} \cdot \frac{n^2-16}{4n+16}$

94. $\frac{6p^2-6p}{p^2+7p-18} \cdot \frac{p^2-81}{3p^2-27p}$

95. $\frac{q^2-2q}{q^2+6q-16} \cdot \frac{q^2-64}{q^2-8q}$

96. $\frac{2r^2-2r}{r^2+4r-5} \cdot \frac{r^2-25}{2r^2-10r}$

Divide Rational Expressions

In the following exercises, divide.

97. $\frac{t-6}{3-t} \div \frac{t^2-9}{t-5}$

98. $\frac{v-5}{11-v} \div \frac{v^2-25}{v-11}$

99. $\frac{10+w}{w-8} \div \frac{100-w^2}{8-w}$

100. $\frac{7+x}{x-6} \div \frac{49-x^2}{x+6}$

101. $\frac{27y^2}{3y-21} \div \frac{3y^2+18}{y^2+13y+42}$

102. $\frac{24z^2}{2z-8} \div \frac{4z-28}{z^2-11z+28}$

103. $\frac{16a^2}{4a+36} \div \frac{4a^2-24a}{a^2+4a-45}$

104. $\frac{24b^2}{2b-4} \div \frac{12b^2+36b}{b^2-11b+18}$

105. $\frac{5c^2+9c+2}{c^2-25} \div \frac{3c^2-14c-5}{c^2+10c+25}$

106. $\frac{2d^2+d-3}{d^2-16} \div \frac{2d^2-9d-18}{d^2-8d+16}$

107. $\frac{6m^2 - 2m - 10}{9 - m^2} \div \frac{6m^2 + 29m - 20}{m^2 - 6m + 9}$

108. $\frac{2n^2 - 3n - 14}{25 - n^2} \div \frac{2n^2 - 13n + 21}{n^2 - 10n + 25}$

109. $\frac{3s^2}{s^2 - 16} \div \frac{s^3 - 4s^2 + 16s}{s^3 - 64}$

110. $\frac{r^2 - 9}{15} \div \frac{r^3 - 27}{5r^2 + 15r + 45}$

111. $\frac{p^3 + q^3}{3p^2 + 3pq + 3q^2} \div \frac{p^2 - q^2}{12}$

112. $\frac{v^3 - 8w^3}{2v^2 + 4vw + 8w^2} \div \frac{v^2 - 4w^2}{4}$

113. $\frac{t^2 - 9}{2t} \div (t^2 - 6t + 9)$

114. $\frac{x^2 + 3x - 10}{4x} \div (2x^2 + 20x + 50)$

115. $\frac{2y^2 - 10yz - 48z^2}{2y - 1} \div (4y^2 - 32yz)$

116. $\frac{2m^2 - 98n^2}{2m + 6} \div (m^2 - 7mn)$

117. $\dfrac{\frac{2a^2 - a - 21}{5a + 20}}{\frac{a^2 + 7a + 12}{a^2 + 8a + 16}}$

118. $\dfrac{\frac{3b^2 + 2b - 8}{12b + 18}}{\frac{3b^2 + 2b - 8}{2b^2 - 7b - 15}}$

119. $\dfrac{\frac{12c^2 - 12}{2c^2 - 3c + 1}}{\frac{4c + 4}{6c^2 - 13c + 5}}$

120. $\dfrac{\frac{4d^2 + 7d - 2}{35d + 10}}{\frac{d^2 - 4}{7d^2 - 12d - 4}}$

121. $\frac{10m^2 + 80m}{3m - 9} \cdot \frac{m^2 + 4m - 21}{m^2 - 9m + 20} \div \frac{5m^2 + 10m}{2m - 10}$

122. $\frac{4n^2 + 32n}{3n + 2} \cdot \frac{3n^2 - n - 2}{n^2 + n - 30} \div \frac{108n^2 - 24n}{n + 6}$

123. $\frac{12p^2 + 3p}{p + 3} \div \frac{p^2 + 2p - 63}{p^2 - p - 12} \cdot \frac{p - 7}{9p^3 - 9p^2}$

124. $\frac{6q + 3}{9q^2 - 9q} \div \frac{q^2 + 14q + 33}{q^2 + 4q - 5} \cdot \frac{4q^2 + 12q}{12q + 6}$

Everyday Math

125. Probability The director of large company is interviewing applicants for two identical jobs. If $w =$ the number of women applicants and $m =$ the number of men applicants, then the probability that two women are selected for the jobs is $\frac{w}{w + m} \cdot \frac{w - 1}{w + m - 1}$.

ⓐ Simplify the probability by multiplying the two rational expressions.

ⓑ Find the probability that two women are selected when $w = 5$ and $m = 10$.

126. Area of a triangle The area of a triangle with base b and height h is $\frac{bh}{2}$. If the triangle is stretched to make a new triangle with base and height three times as much as in the original triangle, the area is $\frac{9bh}{2}$. Calculate how the area of the new triangle compares to the area of the original triangle by dividing $\frac{9bh}{2}$ by $\frac{bh}{2}$.

Writing Exercises

127.

ⓐ Multiply $\frac{7}{4} \cdot \frac{9}{10}$ and explain all your steps.

ⓑ Multiply $\frac{n}{n-3} \cdot \frac{9}{n+3}$ and explain all your steps.

ⓒ Evaluate your answer to part (b) when $n = 7$. Did you get the same answer you got in part (a)? Why or why not?

128.

ⓐ Divide $\frac{24}{5} \div 6$ and explain all your steps.

ⓑ Divide $\frac{x^2 - 1}{x} \div (x + 1)$ and explain all your steps.

ⓒ Evaluate your answer to part (b) when $x = 5$. Did you get the same answer you got in part (a)? Why or why not?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
multiply rational expressions.			
divide rational expressions.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

8.3 Add and Subtract Rational Expressions with a Common Denominator

Learning Objectives

By the end of this section, you will be able to:

- Add rational expressions with a common denominator
- Subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Add: $\frac{y}{3}+\frac{9}{3}$.
   If you missed this problem, review **Example 1.77**.
2. Subtract: $\frac{10}{x}-\frac{2}{x}$.
   If you missed this problem, review **Example 1.79**.
3. Factor completely: $8n^5-20n^3$.
   If you missed this problem, review **Example 7.59**.
4. Factor completely: $45a^3-5ab^2$.
   If you missed this problem, review **Example 7.62**.

Add Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

Rational Expression Addition

If $p,\ q,$ and r are polynomials where $r \neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r}$$

To add rational expressions with a common denominator, add the numerators and place the sum over the common denominator.

We will add two numerical fractions first, to remind us of how this is done.

EXAMPLE 8.30

Add: $\frac{5}{18}+\frac{7}{18}$.

Solution

$$\frac{5}{18}+\frac{7}{18}$$

The fractions have a common denominator, so add the numerators and place the sum over the common denominator.

$$\frac{5+7}{18}$$

Add in the numerator.

$$\frac{12}{18}$$

Factor the numerator and denominator to show the common factors.

$$\frac{6\cdot 2}{6\cdot 3}$$

Remove common factors.

$$\frac{\cancel{6}\cdot 2}{\cancel{6}\cdot 3}$$

Simplify.

$$\frac{2}{3}$$

TRY IT : : 8.59 Add: $\frac{7}{16}+\frac{5}{16}$.

TRY IT : : 8.60 Add: $\frac{3}{10}+\frac{1}{10}$.

Remember, we do not allow values that would make the denominator zero. What value of y should be excluded in the next example?

EXAMPLE 8.31

Add: $\frac{3y}{4y-3}+\frac{7}{4y-3}$.

Solution

$$\frac{3y}{4y-3}+\frac{7}{4y-3}$$

The fractions have a common denominator, so add the numerators and place the sum over the common denominator.

$$\frac{3y+7}{4y-3}$$

The numerator and denominator cannot be factored. The fraction is simplified.

TRY IT : : 8.61 Add: $\frac{5x}{2x+3}+\frac{2}{2x+3}$.

TRY IT : : 8.62 Add: $\frac{x}{x-2}+\frac{1}{x-2}$.

EXAMPLE 8.32

Add: $\frac{7x+12}{x+3}+\frac{x^2}{x+3}$.

Solution

	$\frac{7x+12}{x+3}+\frac{x^2}{x+3}$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{7x+12+x^2}{x+3}$
Write the degrees in descending order.	$\frac{x^2+7x+12}{x+3}$
Factor the numerator.	$\frac{(x+3)(x+4)}{x+3}$
Simplify by removing common factors.	$\frac{\cancel{(x+3)}(x+4)}{\cancel{x+3}}$
Simplify.	$x+4$

TRY IT : : 8.63 Add: $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

TRY IT : : 8.64 Add: $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

Subtract Rational Expressions with a Common Denominator

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator.

Rational Expression Subtraction

If $p, q,$ and r are polynomials where $r \neq 0$, then

$$\frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

To subtract rational expressions, subtract the numerators and place the difference over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

EXAMPLE 8.33

Subtract: $\frac{n^2}{n-10}-\frac{100}{n-10}$.

Solution

$$\frac{n^2}{n-10} - \frac{100}{n-10}$$

The fractions have a common denominator, so subtract the numerators and place the diffe ence over the common denominator.

$$\frac{n^2 - 100}{n-10}$$

Factor the numerator.

$$\frac{(n-10)(n+10)}{n-10}$$

Simplify by removing common factors.

$$\frac{\cancel{(n-10)}(n+10)}{\cancel{n-10}}$$

Simplify.

$$n + 10$$

TRY IT : : 8.65 Subtract: $\frac{x^2}{x+3} - \frac{9}{x+3}$.

TRY IT : : 8.66 Subtract: $\frac{4x^2}{2x-5} - \frac{25}{2x-5}$.

Be careful of the signs when you subtract a binomial!

EXAMPLE 8.34

Subtract: $\frac{y^2}{y-6} - \frac{2y+24}{y-6}$.

Solution

$$\frac{y^2}{y-6} - \frac{2y+24}{y-6}$$

The fractions have a common denominator, so subtract the numerators and place the diffe ence over the common denominator.

$$\frac{y^2 - (2y+24)}{y-6}$$

Distribute the sign in the numerator.

$$\frac{y^2 - 2y - 24}{y-6}$$

Factor the numerator.

$$\frac{(y-6)(y+4)}{y-6}$$

Remove common factors.

$$\frac{\cancel{(y-6)}(y+4)}{\cancel{y-6}}$$

Simplify.

$$y + 4$$

TRY IT : : 8.67 Subtract: $\frac{n^2}{n-4} - \frac{n+12}{n-4}$.

TRY IT : : 8.68 Subtract: $\dfrac{y^2}{y-1} - \dfrac{9y-8}{y-1}$.

EXAMPLE 8.35

Subtract: $\dfrac{5x^2-7x+3}{x^2-3x+18} - \dfrac{4x^2+x-9}{x^2-3x+18}$.

Solution

	$\dfrac{5x^2-7x+3}{x^2-3x+18} - \dfrac{4x^2+x-9}{x^2-3x+18}$
Subtract the numerators and place the diffe ence over the common denominator.	$\dfrac{5x^2-7x+3-\left(4x^2+x-9\right)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\dfrac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$
Combine like terms.	$\dfrac{x^2-8x+12}{x^2-3x-18}$
Factor the numerator and the denominator.	$\dfrac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify by removing common factors.	$\dfrac{(x-2)\cancel{(x-6)}}{(x+3)\cancel{(x-6)}}$
Simplify.	$\dfrac{(x-2)}{(x+3)}$

TRY IT : : 8.69 Subtract: $\dfrac{4x^2-11x+8}{x^2-3x+2} - \dfrac{3x^2+x-3}{x^2-3x+2}$.

TRY IT : : 8.70 Subtract: $\dfrac{6x^2-x+20}{x^2-81} - \dfrac{5x^2+11x-7}{x^2-81}$.

Add and Subtract Rational Expressions whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let's see how this works.

	$\dfrac{7}{d} + \dfrac{5}{-d}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\dfrac{7}{d} + \dfrac{(-1)5}{(-1)(-d)}$

The denominators are the same.	$\frac{7}{d}+\frac{-5}{d}$
Simplify.	$\frac{2}{d}$

EXAMPLE 8.36

Add: $\dfrac{4u-1}{3u-1}+\dfrac{u}{1-3u}$.

Solution

	$\frac{4u-1}{3u-1}+\frac{u}{1-3u}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{4u-1}{3u-1}+\frac{(-1)u}{(-1)(1-3u)}$
Simplify the second fraction.	$\frac{4u-1}{3u-1}+\frac{-u}{3u-1}$
The denominators are the same. Add the numerators.	$\frac{4u-1-u}{3u-1}$
Simplify.	$\frac{3u-1}{3u-1}$
Simplify.	1

TRY IT : : 8.71 Add: $\dfrac{8x-15}{2x-5}+\dfrac{2x}{5-2x}$.

TRY IT : : 8.72 Add: $\dfrac{6y^2+7y-10}{4y-7}+\dfrac{2y^2+2y+11}{7-4y}$.

EXAMPLE 8.37

Subtract: $\dfrac{m^2-6m}{m^2-1}-\dfrac{3m+2}{1-m^2}$.

Solution

	$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1}-\frac{-1(3m+2)}{-1(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1}-\frac{-3m-2}{m^2-1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$

Distribute.	$\dfrac{m^2-6m+3m+2}{m^2-1}$
Combine like terms.	$\dfrac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\dfrac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify by removing common factors.	$\dfrac{\cancel{(m-1)}(m-2)}{\cancel{(m-1)}(m+1)}$
Simplify.	$\dfrac{m-2}{m+1}$

TRY IT : : 8.73 Subtract: $\dfrac{y^2-5y}{y^2-4}-\dfrac{6y-6}{4-y^2}$.

TRY IT : : 8.74 Subtract: $\dfrac{2n^2+8n-1}{n^2-1}-\dfrac{n^2-7n-1}{1-n^2}$.

8.3 EXERCISES

Practice Makes Perfect

Add Rational Expressions with a Common Denominator

In the following exercises, add.

129. $\frac{2}{15}+\frac{7}{15}$

130. $\frac{4}{21}+\frac{3}{21}$

131. $\frac{7}{24}+\frac{11}{24}$

132. $\frac{7}{36}+\frac{13}{36}$

133. $\frac{3a}{a-b}+\frac{1}{a-b}$

134. $\frac{3c}{4c-5}+\frac{5}{4c-5}$

135. $\frac{d}{d+8}+\frac{5}{d+8}$

136. $\frac{7m}{2m+n}+\frac{4}{2m+n}$

137. $\frac{p^2+10p}{p+2}+\frac{16}{p+2}$

138. $\frac{q^2+12q}{q+3}+\frac{27}{q+3}$

139. $\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$

140. $\frac{3s^2}{3s-2}+\frac{13s-10}{3s-2}$

141. $\frac{8t^2}{t+4}+\frac{32t}{t+4}$

142. $\frac{6v^2}{v+5}+\frac{30v}{v+5}$

143. $\frac{2w^2}{w^2-16}+\frac{8w}{w^2-16}$

144. $\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

Subtract Rational Expressions with a Common Denominator

In the following exercises, subtract.

145. $\frac{y^2}{y+8}-\frac{64}{y+8}$

146. $\frac{z^2}{z+2}-\frac{4}{z+2}$

147. $\frac{9a^2}{3a-7}-\frac{49}{3a-7}$

148. $\frac{25b^2}{5b-6}-\frac{36}{5b-6}$

149. $\frac{c^2}{c-8}-\frac{6c+16}{c-8}$

150. $\frac{d^2}{d-9}-\frac{6d+27}{d-9}$

151. $\frac{3m^2}{6m-30}-\frac{21m-30}{6m-30}$

152. $\frac{2n^2}{4n-32}-\frac{30n-16}{4n-32}$

153. $\frac{6p^2+3p+4}{p^2+4p-5}-\frac{5p^2+p+7}{p^2+4p-5}$

154. $\frac{5q^2+3q-9}{q^2+6q+8}-\frac{4q^2+9q+7}{q^2+6q+8}$

155. $\frac{5r^2+7r-33}{r^2-49}-\frac{4r^2-5r-30}{r^2-49}$

156. $\frac{7t^2-t-4}{t^2-25}-\frac{6t^2+2t-1}{t^2-25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add.

157. $\frac{10v}{2v-1}+\frac{2v+4}{1-2v}$

158. $\frac{20w}{5w-2}+\frac{5w+6}{2-5w}$

159. $\frac{10x^2+16x-7}{8x-3}+\frac{2x^2+3x-1}{3-8x}$

160. $\dfrac{6y^2+2y-11}{3y-7}+\dfrac{3y^2-3y+17}{7-3y}$

In the following exercises, subtract.

161. $\dfrac{z^2+6z}{z^2-25}-\dfrac{3z+20}{25-z^2}$

162. $\dfrac{a^2+3a}{a^2-9}-\dfrac{3a-27}{9-a^2}$

163. $\dfrac{2b^2+30b-13}{b^2-49}-\dfrac{2b^2-5b-8}{49-b^2}$

164. $\dfrac{c^2+5c-10}{c^2-16}-\dfrac{c^2-8c-10}{16-c^2}$

Everyday Math

165. Sarah ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. If r represents Sarah's speed when she ran, then her running time is modeled by the expression $\frac{8}{r}$ and her biking time is modeled by the expression $\frac{24}{r+4}$. Add the rational expressions $\frac{8}{r}+\frac{24}{r+4}$ to get an expression for the total amount of time Sarah ran and biked.

166. If Pete can paint a wall in p hours, then in one hour he can paint $\frac{1}{p}$ of the wall. It would take Penelope 3 hours longer than Pete to paint the wall, so in one hour she can paint $\frac{1}{p+3}$ of the wall. Add the rational expressions $\frac{1}{p}+\frac{1}{p+3}$ to get an expression for the part of the wall Pete and Penelope would paint in one hour if they worked together.

Writing Exercises

167. Donald thinks that $\frac{3}{x}+\frac{4}{x}$ is $\frac{7}{2x}$. Is Donald correct? Explain.

168. Explain how you find the Least Common Denominator of x^2+5x+4 and x^2-16.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
add rational expressions with a common denominator.			
subtract rational expressions with a common denominator.			
add and subtract rational expressions whose denominators are opposites.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

8.4 Add and Subtract Rational Expressions with Unlike Denominators

Learning Objectives

By the end of this section, you will be able to:

- Find the least common denominator of rational expressions
- Find equivalent rational expressions
- Add rational expressions with different denominators
- Subtract rational expressions with different denominators

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Add: $\frac{7}{10} + \frac{8}{15}$.
   If you missed this problem, review **Example 1.81**.
2. Subtract: $6(2x + 1) - 4(x - 5)$.
   If you missed this problem, review **Example 1.139**.
3. Find the Greatest Common Factor of $9x^2y^3$ and $12xy^5$.
   If you missed this problem, review **Example 7.3**.
4. Factor completely $-48n - 12$.
   If you missed this problem, review **Example 7.11**.

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at the example $\frac{7}{12} + \frac{5}{18}$ from **Foundations**. Since the denominators are not the same, the first step was to find the least common denominator (LCD). Remember, the LCD is the least common multiple of the denominators. It is the smallest number we can use as a common denominator.

To find the LCD of 12 and 18, we factored each number into primes, lining up any common primes in columns. Then we "brought down" one prime from each column. Finally, we multiplied the factors to find the LCD.

$$\begin{array}{r} 12 = 2 \cdot 2 \cdot 3 \\ \underline{18 = 2 \cdot 3 \cdot 3} \\ \text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3 \\ \text{LCD} = 36 \end{array}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

HOW TO : : FIND THE LEAST COMMON DENOMINATOR OF RATIONAL EXPRESSIONS.

Step 1. Factor each expression completely.

Step 2. List the factors of each expression. Match factors vertically when possible.

Step 3. Bring down the columns.

Step 4. Multiply the factors.

Remember, we always exclude values that would make the denominator zero. What values of *x* should we exclude in this next example?

EXAMPLE 8.38

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.

Solution

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.

Factor each expression completely, lining up common factors. Bring down the columns.	$$\begin{array}{rl} x^2-2x-3 = & (x+1)(x-2) \\ x^2+4x+3 = & (x+1)\quad(x+3) \\ \hline \text{LCD} = & (x+1)(x-2)(x+3) \end{array}$$
Multiply the factors.	The LCD is $(x+1)(x-3)(x+3)$.

> **TRY IT : :** 8.75 Find the LCD for $\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$.

> **TRY IT : :** 8.76 Find the LCD for $\frac{x}{x^2+8x+15}, \frac{5}{x^2+9x+18}$.

Find Equivalent Rational Expressions

When we add numerical fractions, once we find the LCD, we rewrite each fraction as an equivalent fraction with the LCD.

$$\frac{7}{12}+\frac{5}{18} \qquad \begin{array}{rl} 12 = & 2\cdot 2\cdot 3 \\ 18 = & 2\cdot \quad\ \ 3\cdot 3 \\ \hline \text{LCD} = & 2\cdot 2\cdot 3\cdot 3 \\ \text{LCD} = & 36 \end{array}$$

$$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$$

$$\frac{21}{36}+\frac{10}{36}$$

We will do the same thing for rational expressions.

EXAMPLE 8.39

Rewrite as equivalent rational expressions with denominator $(x+1)(x-3)(x+3)$: $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.

Solution

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x+1)(x-3)}, \frac{3x}{(x+1)(x+3)}$
Find the LCD. $\begin{array}{rl} x^2-2x-3 = & (x+1)(x-3)\quad\quad \\ x^2+4x+3 = & (x+1)\quad\quad(x+3) \\ \hline \text{LCD} \quad = & (x+1)(x-3)(x+3) \end{array}$	
Multiply each denominator by the 'missing' factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}, \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

TRY IT : : 8.77 Rewrite as equivalent rational expressions with denominator $(x+3)(x-4)(x+4)$: $\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$.

TRY IT : : 8.78 Rewrite as equivalent rational expressions with denominator $(x+3)(x+5)(x+6)$: $\frac{x}{x^2+8x+15}, \frac{5}{x^2+9x+18}$.

Add Rational Expressions with Different Denominators

Now we have all the steps we need to add rational expressions with different denominators. As we have done previously, we will do one example of adding numerical fractions first.

EXAMPLE 8.40

Add: $\frac{7}{12}+\frac{5}{18}$.

Solution

		$\frac{7}{12}+\frac{5}{18}$
Find the LCD of 12 and 18.	$\begin{array}{l} 12 = 2\cdot2\cdot3 \\ 18 = 2\cdot3\cdot3 \\ \hline \text{LCD} = 2\cdot2\cdot3\cdot3 \\ \text{LCD} = 36 \end{array}$	
Rewrite each fraction as an equivalent fraction with the LCD.		$\frac{7\cdot3}{12\cdot3}+\frac{5\cdot2}{18\cdot2}$
Add the fractions.		$\frac{21}{36}+\frac{10}{36}$
The fraction cannot be simplified.		$\frac{31}{36}$

TRY IT : : 8.79 Add: $\frac{11}{30}+\frac{7}{12}$.

TRY IT : : 8.80 Add: $\frac{3}{8}+\frac{9}{20}$.

Now we will add rational expressions whose denominators are monomials.

EXAMPLE 8.41

Add: $\frac{5}{12x^2y}+\frac{4}{21xy^2}$.

Solution

	$\frac{5}{12x^2y}+\frac{4}{21xy^2}$
Find the LCD of $12x^2y$ and $21xy^2$.	$\begin{array}{l} 12x^2y = 2\cdot2\cdot3\cdotx\cdot x\cdot y \\ 21xy^2 = 3\cdot7\cdot x\cdot y\cdot y \\ \hline \text{LCD} = 2\cdot2\cdot3\cdot7\cdot x\cdot x\cdot y\cdot y \\ \text{LCD} = 84x^2y^2 \end{array}$
	$\frac{5}{12x^2y}+\frac{4}{21xy^2}$
Rewrite each rational expression as an equivalent fraction with the LCD.	$\frac{5\cdot 7y}{12x^2y\cdot 7y}+\frac{4\cdot 4x}{21xy^2\cdot 4x}$
Simplify.	$\frac{35y}{84x^2y^2}+\frac{16x}{84x^2y^2}$
Add the rational expressions.	$\frac{16x+35y}{84x^2y^2}$
There are no factors common to the numerator and denominator. The fraction cannot be simplified.	

TRY IT : : 8.81 Add: $\frac{2}{15a^2b}+\frac{5}{6ab^2}$.

TRY IT : : 8.82 Add: $\frac{5}{16c}+\frac{3}{8cd^2}$.

Now we are ready to tackle polynomial denominators.

EXAMPLE 8.42 HOW TO ADD RATIONAL EXPRESSIONS WITH DIFFERENT DENOMINATORS

Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Solution

Step 1. Determine if the expressions have a common denominator. • Yes – Go to step 2. • No – Rewrite each rational expression with the LCD. ○ Find the LCD. ○ Rewrite each rational expression as an equivalent rational expression with the LCD.	No Find the LCD of $(x-3)$, $(x-2)$ Change into equivalent rational expressions with the LCD, $(x-3)(x-2)$. Keep the denominators factored!	$\begin{array}{ll} x-3 & :(x-3) \\ x-2 & :(x-2) \\ \hline \text{LCD} & :(x-3)(x-2) \end{array}$ $\frac{3}{x-3}+\frac{2}{x-2}$ $\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$ $\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$

Step 2. Add the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$ $\frac{5x-12}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	

TRY IT : : 8.83 Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

TRY IT : : 8.84 Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

The steps to use to add rational expressions are summarized in the following procedure box.

HOW TO : : ADD RATIONAL EXPRESSIONS.

Step 1. Determine if the expressions have a common denominator.
Yes – go to step 2.
No – Rewrite each rational expression with the LCD.
Find the LCD.
Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 2. Add the rational expressions.

Step 3. Simplify, if possible.

EXAMPLE 8.43

Add: $\frac{2a}{2ab+b^2}+\frac{3a}{4a^2-b^2}$.

Solution

	$\frac{2a}{2ab+b^2}+\frac{3a}{4a^2-b^2}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{aligned} 2ab+b^2 &= b(2a+b) \\ 4a^2-b^2 &= (2a+b)(2a-b) \\ \hline \text{LCD} &= b(2a+b)(2a-b) \end{aligned}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2a(2a-b)}{b(2a+b)(2a-b)}+\frac{3a\cdot b}{(2a+b)(2a-b)\cdot b}$
Simplify the numerators.	$\frac{4a^2-2ab}{b(2a+b)(2a-b)}+\frac{3ab}{b(2a+b)(2a-b)}$
Add the rational expressions.	$\frac{4a^2-2ab+3ab}{b(2a+b)(2a-b)}$
Simplify the numerator.	$\frac{4a^2+ab}{b(2a+b)(2a-b)}$
Factor the numerator.	$\frac{a(4a+b)}{b(2a+b)(2a-b)}$
There are no factors common to the numerator and denominator. The fraction cannot be simplified.	

TRY IT : : 8.85 Add: $\frac{5x}{xy-y^2}+\frac{2x}{x^2+y^2}$.

TRY IT : : 8.86 Add: $\frac{7}{2m+6}+\frac{4}{m^2+4m+3}$.

Avoid the temptation to simplify too soon! In the example above, we must leave the first rational expression as $\frac{2a(2a-b)}{b(2a+b)(2a-b)}$ to be able to add it to $\frac{3a\cdot b}{(2a+b)(2a-b)\cdot b}$. Simplify only after you have combined the numerators.

EXAMPLE 8.44

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Solution

	$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{aligned} x^2-2x-3 &= (x+1)(x-3) \\ x^2+4x+3 &= (x+1)\qquad(x+3) \\ \hline \text{LCD} &= (x+1)(x-3)(x+3) \end{aligned}$	
Rewrite each rational expression as an equivalent fraction with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$

Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)} + \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$\frac{8x+24+3x^2+9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x^2+24}{(x+1)(x-3)(x+3)}$
The numerator is prime, so there are no common factors.	

TRY IT : : 8.87 Add: $\frac{1}{m^2-m-2}+\frac{5m}{m^2+3m+2}$.

TRY IT : : 8.88 Add: $\frac{2n}{n^2-3n-10}+\frac{6}{n^2+5n+6}$.

Subtract Rational Expressions with Different Denominators

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

EXAMPLE 8.45 HOW TO SUBTRACT RATIONAL EXPRESSIONS WITH DIFFERENT DENOMINATORS

Subtract: $\frac{x}{x-3}-\frac{x-2}{x+3}$.

Solution

Step 1. Determine if the expressions have a common denominator. • Yes – Go to step 2. • No – Rewrite each rational expression with the LCD. ○ Find the LCD. ○ Rewrite each rational expression as an equivalent rational expression with the LCD.	No Find the LCD of $(x-3)$, $(x+3)$. Change into equivalent fractions with the LCD, $(x-3)(x+3)$. Keep the denominators factored!	$x-3$: $(x-3)$ $x+3$: $\quad(x+3)$ LCD: $(x-3)(x+3)$ $\frac{x}{x-3}-\frac{x-2}{x+3}$ $\frac{x(x+3)}{(x-3)(x+3)}-\frac{(x-2)(x-3)}{(x+3)(x-3)}$ $\frac{x^2+3x}{(x-3)(x+3)}-\frac{x^2-5x+6}{(x-3)(x+3)}$
Step 2. Subtract the rational expressions.	Subtract the numerators and place the difference over the common denominator. Be careful with the signs!	$\frac{x^2+3x-(x^2-5x+6)}{(x-3)(x+3)}$ $\frac{x^2+3x-x^2+5x-6}{(x-3)(x+3)}$ $\frac{8x-6}{(x-3)(x+3)}$
Step 3. Simplify, if possible.	The numerator and denominator have no factors in common. The answer is simplified.	$\frac{2(4x-3)}{(x-3)(x+3)}$

TRY IT : : 8.89 Subtract: $\frac{y}{y+4}-\frac{y-2}{y-5}$.

TRY IT : : 8.90 Subtract: $\frac{z+3}{z+2} - \frac{z}{z+3}$.

The steps to take to subtract rational expressions are listed below.

HOW TO : : SUBTRACT RATIONAL EXPRESSIONS.

Step 1. Determine if they have a common denominator.
Yes – go to step 2.
No – Rewrite each rational expression with the LCD.
Find the LCD.
Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 2. Subtract the rational expressions.

Step 3. Simplify, if possible.

EXAMPLE 8.46

Subtract: $\frac{8y}{y^2-16} - \frac{4}{y-4}$.

Solution

	$\frac{8y}{y^2-16} - \frac{4}{y-4}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{array}{rl} y^2-16 &= (y-4)(y+4) \\ y-4 &= y-4 \\ \hline \text{LCD} &= (y-4)(y+4) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)} - \frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)} - \frac{4y+16}{(y-4)(y+4)}$
Subtract the rational expressions.	$\frac{8y-4y-16}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{4y-16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$
Remove common factors.	$\frac{4\cancel{(y-4)}}{\cancel{(y-4)}(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

TRY IT : : 8.91 Subtract: $\frac{2x}{x^2-4} - \frac{1}{x+2}$.

TRY IT : : 8.92 Subtract: $\frac{3}{z+3} - \frac{6z}{z^2 - 9}$.

There are lots of negative signs in the next example. Be extra careful!

EXAMPLE 8.47

Subtract: $\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$.

Solution

	$\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$
Factor the denominator.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{n+3}{2-n}$
Since $n-2$ and $2-n$ are opposites, we will mutliply the second rational expression by $\frac{-1}{-1}$.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{(-1)(n+3)}{(-1)(2-n)}$
Simplify.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)}{(n-2)}$
Do the expressions have a common denominator? No.	
Find the LCD. $\begin{array}{r} n^2+n-6=(n-2)(n+3) \\ n-2=(n-2) \\ \hline \text{LCD}=(n-2)(n+3) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)(n+3)}{(n-2)(n+3)}$
Simplify the numerators.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{n^2+6n+9}{(n-2)(n+3)}$
Simplify the rational expressions.	$\frac{-3n-9+n^2+6n+9}{(n-2)(n+3)}$
Somplify the numerator.	$\frac{n^2+3n}{(n-2)(n+3)}$
Factor the numerator to look for common factors.	$\frac{n\cancel{(n+3)}}{(n-2)\cancel{(n+3)}}$
Simplify.	$\frac{n}{(n-2)}$

TRY IT : : 8.93 Subtract: $\frac{3x-1}{x^2-5x-6} - \frac{2}{6-x}$.

TRY IT : : 8.94 Subtract: $\frac{-2y-2}{y^2+2y-8} - \frac{y-1}{2-y}$.

When one expression is not in fraction form, we can write it as a fraction with denominator 1.

EXAMPLE 8.48

Subtract: $\frac{5c+4}{c-2} - 3$.

Solution

	$\frac{5c+4}{c-2} - 3$
Write 3 as $\frac{3}{1}$ to have 2 rational expressions.	$\frac{5c+4}{c-2} - \frac{3}{1}$
Do the rational expressions have a common denominator? No.	
Find the LCD of $c-2$ and 1. LCD = $c-2$.	
Rewrite $\frac{3}{1}$ as an equivalent rational expression with the LCD.	$\frac{5c+4}{c-2} - \frac{3(c-2)}{1(c-2)}$
Simplify.	$\frac{5c+4}{c-2} - \frac{3c-6}{c-2}$
Subtract the rational expressions.	$\frac{5c+4-(3c-6)}{c-2}$
Simplify.	$\frac{2c+10}{c-2}$
Factor to check for common factors.	$\frac{2(c+5)}{c-2}$
There are no common factors; the rational expression is simplified.	

TRY IT : : 8.95 Subtract: $\frac{2x+1}{x-7} - 3$.

TRY IT : : 8.96 Subtract: $\frac{4y+3}{2y-1} - 5$.

HOW TO : : ADD OR SUBTRACT RATIONAL EXPRESSIONS.

Step 1. Determine if the expressions have a common denominator.
Yes – go to step 2.
No – Rewrite each rational expression with the LCD.
Find the LCD.
Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 2. Add or subtract the rational expressions.

Step 3. Simplify, if possible.

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example we will start by factoring all three denominators to find their LCD.

EXAMPLE 8.49

Simplify: $\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$.

Solution

	$\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $\begin{array}{l} u-1 = \quad u-1 \\ u \quad\;\; = u \\ \underline{u^2-u = u(u-1)} \\ \text{LCD} = u(u-1) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2u \cdot u}{(u-1)u}+\frac{1\cdot(u-1)}{u\cdot(u-1)}-\frac{2u-1}{u(u-1)}$
	$\frac{2u^2}{(u-1)u}+\frac{u-1}{u\cdot(u-1)}-\frac{2u-1}{u(u-1)}$
Write as one rational expression.	$\frac{2u^2+u-1-2u+1}{u(u-1)}$
Simplify.	$\frac{2u^2-u}{u(u-1)}$
Factor the numerator, and remove common factors.	$\frac{\cancel{u}(2u-1)}{\cancel{u}(u-1)}$
Simplify.	$\frac{2u-1}{u-1}$

TRY IT : : 8.97 Simplify: $\frac{v}{v+1}+\frac{3}{v-1}-\frac{6}{v^2-1}$.

TRY IT : : 8.98 Simplify: $\frac{3w}{w+2}+\frac{2}{w+7}-\frac{17w+4}{w^2+9w+14}$.

8.4 EXERCISES

Practice Makes Perfect

In the following exercises, find the LCD.

169. $\frac{5}{x^2 - 2x - 8}, \frac{2x}{x^2 - x - 12}$

170. $\frac{8}{y^2 + 12y + 35}, \frac{3y}{y^2 + y - 42}$

171. $\frac{9}{z^2 + 2z - 8}, \frac{4z}{z^2 - 4}$

172. $\frac{6}{a^2 + 14a + 45}, \frac{5a}{a^2 - 81}$

173. $\frac{4}{b^2 + 6b + 9}, \frac{2b}{b^2 - 2b - 15}$

174. $\frac{5}{c^2 - 4c + 4}, \frac{3c}{c^2 - 10c + 16}$

175. $\frac{2}{3d^2 + 14d - 5}, \frac{5d}{3d^2 - 19d + 6}$

176. $\frac{3}{5m^2 - 3m - 2}, \frac{6m}{5m^2 + 17m + 6}$

In the following exercises, write as equivalent rational expressions with the given LCD.

177. $\frac{5}{x^2 - 2x - 8}, \frac{2x}{x^2 - x - 12}$
LCD $(x - 4)(x + 2)(x + 3)$

178. $\frac{8}{y^2 + 12y + 35}, \frac{3y}{y^2 + y - 42}$
LCD $(y + 7)(y + 5)(y - 6)$

179. $\frac{9}{z^2 + 2z - 8}, \frac{4z}{z^2 - 4}$
LCD $(z - 2)(z + 4)(z + 2)$

180. $\frac{6}{a^2 + 14a + 45}, \frac{5a}{a^2 - 81}$
LCD $(a + 9)(a + 5)(a - 9)$

181. $\frac{4}{b^2 + 6b + 9}, \frac{2b}{b^2 - 2b - 15}$
LCD $(b + 3)(b + 3)(b - 5)$

182. $\frac{5}{c^2 - 4c + 4}, \frac{3c}{c^2 - 10c + 10}$
LCD $(c - 2)(c - 2)(c - 8)$

183. $\frac{2}{3d^2 + 14d - 5}, \frac{5d}{3d^2 - 19d + 6}$
LCD $(3d - 1)(d + 5)(d - 6)$

184. $\frac{3}{5m^2 - 3m - 2}, \frac{6m}{5m^2 + 17m + 6}$
LCD $(5m + 2)(m - 1)(m + 3)$

In the following exercises, add.

185. $\frac{5}{24} + \frac{11}{36}$

186. $\frac{7}{30} + \frac{13}{45}$

187. $\frac{9}{20} + \frac{11}{30}$

188. $\frac{8}{27} + \frac{7}{18}$

189. $\frac{7}{10x^2 y} + \frac{4}{15xy^2}$

190. $\frac{1}{12a^3 b^2} + \frac{5}{9a^2 b^3}$

191. $\frac{1}{2m} + \frac{7}{8m^2 n}$

192. $\frac{5}{6p^2 q} + \frac{1}{4p}$

193. $\frac{3}{r + 4} + \frac{2}{r - 5}$

194. $\frac{4}{s - 7} + \frac{5}{s + 3}$

195. $\frac{8}{t + 5} + \frac{6}{t - 5}$

196. $\frac{7}{v + 5} + \frac{9}{v - 5}$

197. $\frac{5}{3w - 2} + \frac{2}{w + 1}$

198. $\frac{4}{2x + 5} + \frac{2}{x - 1}$

199. $\frac{2y}{y + 3} + \frac{3}{y - 1}$

200. $\frac{3z}{z - 2} + \frac{1}{z + 5}$

201. $\frac{5b}{a^2 b - 2a^2} + \frac{2b}{b^2 - 4}$

202. $\frac{4}{cd + 3c} + \frac{1}{d^2 - 9}$

203. $\frac{2m}{3m-3}+\frac{5m}{m^2+3m-4}$

204. $\frac{3}{4n+4}+\frac{6}{n^2-n-2}$

205. $\frac{3}{n^2+3n-18}+\frac{4n}{n^2+8n+12}$

206. $\frac{6}{q^2-3q-10}+\frac{5q}{q^2-8q+15}$

207. $\frac{3r}{r^2+7r+6}+\frac{9}{r^2+4r+3}$

208. $\frac{2s}{s^2+2s-8}+\frac{4}{s^2+3s-10}$

In the following exercises, subtract.

209. $\frac{t}{t-6}-\frac{t-2}{t+6}$

210. $\frac{v}{v-3}-\frac{v-6}{v+1}$

211. $\frac{w+2}{w+4}-\frac{w}{w-2}$

212. $\frac{x-3}{x+6}-\frac{x}{x+3}$

213. $\frac{y-4}{y+1}-\frac{1}{y+7}$

214. $\frac{z+8}{z-3}-\frac{z}{z-2}$

215. $\frac{5a}{a+3}-\frac{a+2}{a+6}$

216. $\frac{3b}{b-2}-\frac{b-6}{b-8}$

217. $\frac{6c}{c^2-25}-\frac{3}{c+5}$

218. $\frac{4d}{d^2-81}-\frac{2}{d+9}$

219. $\frac{6}{m+6}-\frac{12m}{m^2-36}$

220. $\frac{4}{n+4}-\frac{8n}{n^2-16}$

221. $\frac{-9p-17}{p^2-4p-21}-\frac{p+1}{7-p}$

222. $\frac{7q+8}{q^2-2q-24}-\frac{q+2}{4-q}$

223. $\frac{-2r-16}{r^2+6r-16}-\frac{5}{2-r}$

224. $\frac{2t-30}{t^2+6t-27}-\frac{2}{3-t}$

225. $\frac{5v-2}{v+3}-4$

226. $\frac{6w+5}{w-1}+2$

227. $\frac{2x+7}{10x-1}+3$

228. $\frac{8y-4}{5y+2}-6$

In the following exercises, add and subtract.

229. $\frac{5a}{a-2}+\frac{9}{a}-\frac{2a+18}{a^2-2a}$

230. $\frac{2b}{b-5}+\frac{3}{2b}-\frac{2b-15}{2b^2-10b}$

231. $\frac{c}{c+2}+\frac{5}{c-2}-\frac{11c}{c^2-4}$

232. $\frac{6d}{d-5}+\frac{1}{d+4}-\frac{7d-5}{d^2-d-20}$

In the following exercises, simplify.

233. $\frac{6a}{3ab+b^2}+\frac{3a}{9a^2-b^2}$

234. $\frac{2c}{2c+10}+\frac{7c}{c^2+9c+20}$

235. $\frac{6d}{d^2-64}-\frac{3}{d-8}$

236. $\frac{5}{n+7}-\frac{10n}{n^2-49}$

237. $\frac{4m}{m^2+6m-7}+\frac{2}{m^2+10m+21}$

238. $\frac{3p}{p^2+4p-12}+\frac{1}{p^2+p-30}$

239. $\frac{-5n-5}{n^2+n-6}+\frac{n+1}{2-n}$

240. $\frac{-4b-24}{b^2+b-30}+\frac{b+7}{5-b}$

241. $\frac{7}{15p}+\frac{5}{18pq}$

242. $\frac{3}{20a^2}+\frac{11}{12ab^2}$

243. $\frac{4}{x-2}+\frac{3}{x+5}$

244. $\frac{6}{m+4}+\frac{9}{m-8}$

245. $\frac{2q+7}{y+4} - 2$

246. $\frac{3y-1}{y+4} - 2$

247. $\frac{z+2}{z-5} - \frac{z}{z+1}$

248. $\frac{t}{t-5} - \frac{t-1}{t+5}$

249. $\frac{3d}{d+2} + \frac{4}{d} - \frac{d+8}{d^2+2d}$

250. $\frac{2q}{q+5} + \frac{3}{q-3} - \frac{13q+15}{q^2+2q-15}$

Everyday Math

251. Decorating cupcakes Victoria can decorate an order of cupcakes for a wedding in t hours, so in 1 hour she can decorate $\frac{1}{t}$ of the cupcakes. It would take her sister 3 hours longer to decorate the same order of cupcakes, so in 1 hour she can decorate $\frac{1}{t+3}$ of the cupcakes.

ⓐ Find the fraction of the decorating job that Victoria and her sister, working together, would complete in one hour by adding the rational expressions $\frac{1}{t} + \frac{1}{t+3}$.

ⓑ Evaluate your answer to part (a) when $t = 5$.

252. Kayaking When Trina kayaks upriver, it takes her $\frac{5}{3-c}$ hours to go 5 miles, where c is the speed of the river current. It takes her $\frac{5}{3+c}$ hours to kayak 5 miles down the river.

ⓐ Find an expression for the number of hours it would take Trina to kayak 5 miles up the river and then return by adding $\frac{5}{3-c} + \frac{5}{3+c}$.

ⓑ Evaluate your answer to part (a) when $c = 1$ to find the number of hours it would take Trina if the speed of the river current is 1 mile per hour.

Writing Exercises

253. Felipe thinks $\frac{1}{x} + \frac{1}{y}$ is $\frac{2}{x+y}$.

ⓐ Choose numerical values for x and y and evaluate $\frac{1}{x} + \frac{1}{y}$.

ⓑ Evaluate $\frac{2}{x+y}$ for the same values of x and y you used in part (a).

ⓒ Explain why Felipe is wrong.

ⓓ Find the correct expression for $\frac{1}{x} + \frac{1}{y}$.

254. Simplify the expression $\frac{4}{n^2+6n+9} - \frac{1}{n^2-9}$ and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the least common denominator of rational expressions.			
find equivalent rational expressions.			
add rational expressions with different denominators.			
subtract rational expressions with different denominators.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

8.5 Simplify Complex Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Simplify: $\dfrac{\frac{3}{5}}{\frac{9}{10}}$.

   If you missed this problem, review **Example 1.72**.

2. Simplify: $\dfrac{1-\frac{1}{3}}{4^2+4\cdot 5}$.

   If you missed this problem, review **Example 1.74**.

Complex fractions are fractions in which the numerator or denominator contains a fraction. In Chapter 1 we simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section we will simplify *complex rational expressions*, which are rational expressions with rational expressions in the numerator or denominator.

Complex Rational Expression

A **complex rational expression** is a rational expression in which the numerator or denominator contains a rational expression.

Here are a few complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}} \qquad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} \qquad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

Simplify a Complex Rational Expression by Writing it as Division

We have already seen this complex rational expression earlier in this chapter.

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}$$

We noted that fraction bars tell us to divide, so rewrote it as the division problem

$$\left(\frac{6x^2-7x+2}{4x-8}\right) \div \left(\frac{2x^2-8x+3}{x^2-5x+6}\right)$$

Then we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify rational expressions. We write it as if we were dividing two fractions.

EXAMPLE 8.50

Simplify: $\dfrac{\frac{4}{y-3}}{\frac{8}{y^2-9}}$.

Solution

$$\dfrac{\frac{4}{y-3}}{\frac{8}{y^2-9}}$$

Rewrite the complex fraction as division.	$\dfrac{4}{y-3} \div \dfrac{8}{y^2-9}$
Rewrite as the product of fir t times the reciprocal of the second.	$\dfrac{4}{y-3} \cdot \dfrac{y^2-9}{8}$
Multiply.	$\dfrac{4\left(y^2-9\right)}{8(y-3)}$
Factor to look for common factors.	$\dfrac{4(y-3)(y+3)}{4 \cdot 2(y-3)}$
Remove common factors.	$\dfrac{\cancel{4}\cancel{(y-3)}(y+3)}{\cancel{4} \cdot 2\cancel{(y-3)}}$
Simplify.	$\dfrac{y+3}{2}$

Are there any value(s) of y that should not be allowed? The simplified rational expression has just a constant in the denominator. But the original complex rational expression had denominators of $y-3$ and y^2-9. This expression would be undefined if $y=3$ or $y=-3$.

TRY IT : : 8.99 Simplify: $\dfrac{\frac{2}{x^2-1}}{\frac{3}{x+1}}$.

TRY IT : : 8.100 Simplify: $\dfrac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$.

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

EXAMPLE 8.51

Simplify: $\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Solution

	$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$$
Simplify the numerator and denominator.	
Find the LCD and add the fractions in the numerator. Find the LCD and add the fractions in the denominator.	$$\frac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$$
Simplify the numerator and denominator.	$$\frac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$$
Simplify the numerator and denominator, again.	$$\frac{\frac{3}{6}}{\frac{1}{6}}$$
Rewrite the complex rational expression as a division problem.	$\frac{3}{6}\div\frac{1}{6}$
Multiply the first times by the reciprocal of the second.	$\frac{3}{6}\cdot\frac{6}{1}$
Simplify.	3

TRY IT : : 8.101 Simplify: $\dfrac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$.

TRY IT : : 8.102 Simplify: $\dfrac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$.

EXAMPLE 8.52 HOW TO SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY WRITING IT AS DIVISION

Simplify: $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Solution

Step 1. Simplify the numerator and denominator.	We will simplify the sum in the numerator and difference in the denominator.	$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
	Find a common denominator and add the fractions in the numerator.	$\dfrac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$
	Find a common denominator and subtract the fractions in the numerator.	$\dfrac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$
	We now have just one rational expression in the numerator and one in the denominator.	$\dfrac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$
Step 2. Rewrite the complex rational expression as a division problem.	We write the numerator divided by the denominator.	$\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$
Step 3. Divide the expressions.	Multiply the first by the reciprocal of the second.	$\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$
	Factor any expressions if possible.	$\dfrac{xy(y+x)}{xy(x-y)(x+y)}$
	Remove common factors.	$\dfrac{\cancel{xy}\cancel{(y+x)}}{\cancel{xy}(x-y)\cancel{(x+y)}}$
	Simplify.	$\dfrac{1}{x-y}$

TRY IT : : 8.103

Simplify: $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$.

TRY IT : : 8.104

Simplify: $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$.

HOW TO : : SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY WRITING IT AS DIVISION.

Step 1. Simplify the numerator and denominator.
Step 2. Rewrite the complex rational expression as a division problem.
Step 3. Divide the expressions.

EXAMPLE 8.53

Simplify: $\dfrac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$.

Solution

	$\dfrac{n - \dfrac{4n}{n+5}}{\dfrac{1}{n+5} + \dfrac{1}{n-5}}$
Simplify the numerator and denominator.	
Find the LCD and add the fractions in the numerator. Find the LCD and add the fractions in the denominator.	$\dfrac{\dfrac{n(n+5)}{1(n+5)} - \dfrac{4n}{n+5}}{\dfrac{1(n-5)}{(n+5)(n-5)} + \dfrac{1(n+5)}{(n-5)(n+5)}}$
Simplify the numerators.	$\dfrac{\dfrac{n^2+5n}{n+5} - \dfrac{4n}{n+5}}{\dfrac{n-5}{(n+5)(n-5)} + \dfrac{n+5}{(n-5)(n+5)}}$
Subtract the rational expressions in the numerator and add in the denominator. Simplify.	$\dfrac{\dfrac{n^2+n}{n+5}}{\dfrac{2n}{(n+5)(n-5)}}$
Rewrite as fraction division.	$\dfrac{n^2+n}{n+5} \div \dfrac{2n}{(n+5)(n-5)}$
Multiply the first times the reciprocal of the second.	$\dfrac{n^2+n}{n+5} \cdot \dfrac{(n+5)(n-5)}{2n}$
Factor any expressions if possible.	$\dfrac{n(n+1)(n+5)(n-5)}{(n+5)2n}$
Remove common factors.	$\dfrac{\cancel{n}(n+1)\cancel{(n+5)}(n-5)}{\cancel{(n+5)}2\cancel{n}}$
Simplify.	$\dfrac{(n+1)(n-5)}{2}$

TRY IT : : 8.105 Simplify: $\dfrac{b - \dfrac{3b}{b+5}}{\dfrac{2}{b+5} + \dfrac{1}{b-5}}$.

TRY IT : : 8.106 Simplify: $\dfrac{1 - \dfrac{3}{c+4}}{\dfrac{1}{c+4} + \dfrac{c}{3}}$.

Simplify a Complex Rational Expression by Using the LCD

We “cleared” the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by LCD of all the rational expressions.

Let’s look at the complex rational expression we simplified one way in Example 8.51. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{\text{LCD}}{\text{LCD}}$ we are multiplying by 1, so the value stays the same.

EXAMPLE 8.54

Simplify: $\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Solution

	$\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
The LCD of all the fractions in the whole expression is 6.	
Clear the fractions by multiplying the numerator and denominator by that LCD.	$\dfrac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$
Distribute.	$\dfrac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$
Simplify.	$\dfrac{2+1}{3-2}$
	$\dfrac{3}{1}$
	3

TRY IT : : 8.107 Simplify: $\dfrac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}$.

TRY IT : : 8.108 Simplify: $\dfrac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}$.

EXAMPLE 8.55 HOW TO SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY USING THE LCD

Simplify: $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Solution

Step 1. Find the LCD of all fractions in the complex rational expression.	The LCD of all the fractions is *xy*.	$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Step 2. Multiply the numerator and denominator by the LCD.	Multiply both the numerator and denominator by *xy*.	$\dfrac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$

Step 3. Simplify the expression.	Distribute.	$\dfrac{xy \cdot \frac{1}{x} + xy \cdot \frac{1}{y}}{xy \cdot \frac{x}{y} - xy \cdot \frac{y}{x}}$
		$\dfrac{y+x}{x^2-y^2}$
	Simplify.	$\dfrac{\cancel{(y+x)}}{(x-y)\cancel{(x+y)}}$
	Remove common factors.	$\dfrac{1}{x-y}$

TRY IT : : 8.109

Simplify: $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}$.

TRY IT : : 8.110

Simplify: $\dfrac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$.

HOW TO : : SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY USING THE LCD.

Step 1. Find the LCD of all fractions in the complex rational expression.
Step 2. Multiply the numerator and denominator by the LCD.
Step 3. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

EXAMPLE 8.56

Simplify: $\dfrac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$.

Solution

	$\dfrac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$
Find the LCD of all fractions in the complex rational expression. The LCD is $(x+6)(x-6)$.	
Multiply the numerator and denominator by the LCD.	$\dfrac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}-\frac{4}{(x+6)(x-6)}\right)}$
Simplify the expression.	

Distribute in the denominator.	$\dfrac{(x+6)(x-6)\dfrac{2}{x+6}}{(x+6)(x-6)\left(\dfrac{4}{x-6}\right)-(x+6)(x-6)\left(\dfrac{4}{(x+6)(x-6)}\right)}$
Simplify.	$\dfrac{\cancel{(x+6)}(x-6)\dfrac{2}{\cancel{x+6}}}{(x+6)\cancel{(x-6)}\left(\dfrac{4}{\cancel{x-6}}\right)-\cancel{(x+6)(x-6)}\left(\dfrac{4}{\cancel{(x+6)(x-6)}}\right)}$
Simplify.	$\dfrac{2(x-6)}{4(x+6)-4}$
To simplify the denominator, distribute and combine like terms.	$\dfrac{2(x-6)}{4x+20}$
Remove common factors.	$\dfrac{\cancel{2}(x-6)}{\cancel{2}(2x+10)}$
Simplify.	$\dfrac{x-6}{2x+10}$
Notice that there are no more factors common to the numerator and denominator.	

TRY IT : : 8.111 Simplify: $\dfrac{\dfrac{3}{x+2}}{\dfrac{5}{x-2}-\dfrac{3}{x^2-4}}$.

TRY IT : : 8.112 Simplify: $\dfrac{\dfrac{2}{x-7}-\dfrac{1}{x+7}}{\dfrac{6}{x+7}-\dfrac{1}{x^2-49}}$.

EXAMPLE 8.57

Simplify: $\dfrac{\dfrac{4}{m^2-7m+12}}{\dfrac{3}{m-3}-\dfrac{2}{m-4}}$.

Solution

	$\dfrac{\dfrac{4}{m^2-7m+12}}{\dfrac{3}{m-3}-\dfrac{2}{m-4}}$
Find the LCD of all fractions in the complex rational expression. The LCD is $(m-3)(m-4)$.	
Multiply the numerator and denominator by the LCD.	$\dfrac{(m-3)(m-4)\dfrac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\dfrac{3}{m-3}-\dfrac{2}{m-4}\right)}$
Simplify.	$\dfrac{\cancel{(m-3)(m-4)}\dfrac{4}{\cancel{(m-3)(m-4)}}}{\cancel{(m-3)}(m-4)\left(\dfrac{3}{\cancel{m-3}}\right)-(m-3)\cancel{(m-4)}\left(\dfrac{2}{\cancel{m-4}}\right)}$

Simplify.	$\dfrac{4}{3(m-4)-2(m-3)}$
Distribute.	$\dfrac{4}{3m-12-2m+6}$
Combine like terms.	$\dfrac{4}{m-6}$

TRY IT : : 8.113

Simplify: $\dfrac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}$.

TRY IT : : 8.114

Simplify: $\dfrac{\frac{4y}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}$.

EXAMPLE 8.58

Simplify: $\dfrac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$.

Solution

	$\dfrac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$
Find the LCD of all fractions in the complex rational expression.	
The LCD is $(y+1)(y-1)$.	
Multiply the numerator and denominator by the LCD.	$\dfrac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$
Distribute in the denominator and simplify.	$\dfrac{\cancel{(y+1)}(y-1)\left(\frac{y}{\cancel{y+1}}\right)}{(y+1)(y-1)(1)+(y+1)\cancel{(y-1)}\left(\frac{1}{\cancel{y-1}}\right)}$
Simplify.	$\dfrac{(y-1)y}{(y+1)(y-1)+(y+1)}$
Simplify the denominator, and leave the numerator factored.	$\dfrac{y(y-1)}{y^2-1+y+1}$
	$\dfrac{y(y-1)}{y^2+y}$
Factor the denominator, and remove factors common with the numerator.	$\dfrac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$
Simplify.	$\dfrac{y-1}{y+1}$

TRY IT : : 8.115

Simplify: $\dfrac{\frac{x}{x+3}}{1+\frac{1}{x+3}}$.

TRY IT : : 8.116

Simplify: $\dfrac{1+\frac{1}{x-1}}{\frac{3}{x+1}}$.

8.5 EXERCISES

Practice Makes Perfect

Simplify a Complex Rational Expression by Writing It as Division

In the following exercises, simplify.

255. $\dfrac{\frac{2a}{a+4}}{\frac{4a^2}{a^2-16}}$

256. $\dfrac{\frac{3b}{b-5}}{\frac{b^2}{b^2-25}}$

257. $\dfrac{\frac{5}{c^2+5c-14}}{\frac{10}{c+7}}$

258. $\dfrac{\frac{8}{d^2+9d+18}}{\frac{12}{d+6}}$

259. $\dfrac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$

260. $\dfrac{\frac{1}{2}+\frac{3}{4}}{\frac{3}{5}+\frac{7}{10}}$

261. $\dfrac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

262. $\dfrac{\frac{1}{2}-\frac{1}{6}}{\frac{2}{3}+\frac{3}{4}}$

263. $\dfrac{\frac{n}{m}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

264. $\dfrac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

265. $\dfrac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

266. $\dfrac{\frac{2}{v}+\frac{2}{w}}{\frac{1}{v^2}-\frac{1}{w^2}}$

267. $\dfrac{x-\frac{2x}{x+3}}{\frac{1}{x+3}+\frac{1}{x-3}}$

268. $\dfrac{y-\frac{2y}{y-4}}{\frac{2}{y-4}-\frac{2}{y+4}}$

269. $\dfrac{2-\frac{2}{a+3}}{\frac{1}{a+3}+\frac{a}{2}}$

270. $\dfrac{4-\frac{4}{b-5}}{\frac{1}{b-5}+\frac{b}{4}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

271. $\dfrac{\frac{1}{3}+\frac{1}{8}}{\frac{1}{4}+\frac{1}{12}}$

272. $\dfrac{\frac{1}{4}+\frac{1}{9}}{\frac{1}{6}+\frac{1}{12}}$

273. $\dfrac{\frac{5}{6}+\frac{2}{9}}{\frac{7}{18}-\frac{1}{3}}$

274. $\dfrac{\frac{1}{6}+\frac{4}{15}}{\frac{3}{5}-\frac{1}{2}}$

275. $\dfrac{\frac{c}{d}+\frac{1}{d}}{\frac{1}{d}-\frac{d}{c}}$

276. $\dfrac{\frac{1}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

277. $\dfrac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p^2}-\frac{1}{q^2}}$

278. $\dfrac{\frac{2}{r}+\frac{2}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

279. $\dfrac{\frac{2}{x+5}}{\frac{3}{x-5}+\frac{1}{x^2-25}}$

280. $\dfrac{\frac{5}{y-4}}{\frac{3}{y+4}+\frac{2}{y^2-16}}$

281. $\dfrac{\frac{5}{z^2-64}+\frac{3}{z+8}}{\frac{1}{z+8}+\frac{2}{z-8}}$

282. $\dfrac{\frac{3}{s+6}+\frac{5}{s-6}}{\frac{1}{s^2-36}+\frac{4}{s+6}}$

283. $\dfrac{\frac{4}{a^2-2a-15}}{\frac{1}{a-5}+\frac{2}{a+3}}$

284. $\dfrac{\frac{5}{b^2-6b-27}}{\frac{3}{b-9}+\frac{1}{b+3}}$

285. $\dfrac{\frac{5}{c+2}-\frac{3}{c+7}}{\frac{5c}{c^2+9c+14}}$

286. $\dfrac{\frac{6}{d-4}-\frac{2}{d+7}}{\frac{2d}{d^2+3d-28}}$

287. $\dfrac{2+\frac{1}{p-3}}{\frac{5}{p-3}}$

288. $\dfrac{\frac{n}{n-2}}{3+\frac{5}{n-2}}$

289. $\dfrac{\frac{m}{m+5}}{4+\frac{1}{m-5}}$

290. $\dfrac{7+\frac{2}{q-2}}{\frac{1}{q+2}}$

Simplify

In the following exercises, use either method.

291. $\dfrac{\frac{3}{4}-\frac{2}{7}}{\frac{1}{2}+\frac{5}{14}}$

292. $\dfrac{\frac{v}{w}+\frac{1}{v}}{\frac{1}{v}-\frac{v}{w}}$

293. $\dfrac{\frac{2}{a+4}}{\frac{1}{a^2-16}}$

294. $\dfrac{\frac{3}{b^2-3b-40}}{\frac{5}{b+5}-\frac{2}{b-8}}$

295. $\dfrac{\frac{3}{m}+\frac{3}{n}}{\frac{1}{m^2}-\frac{1}{n^2}}$

296. $\dfrac{\frac{2}{r-9}}{\frac{1}{r+9}+\frac{3}{r^2-81}}$

297. $\dfrac{x-\frac{3x}{x+2}}{\frac{3}{x+2}+\frac{3}{x-2}}$

298. $\dfrac{\frac{y}{y+3}}{2+\frac{1}{y-3}}$

Everyday Math

299. Electronics The resistance of a circuit formed by connecting two resistors in parallel is $\dfrac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$.

ⓐ Simplify the complex fraction $\dfrac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$.

ⓑ Find the resistance of the circuit when $R_1 = 8$ and $R_2 = 12$.

300. Ironing Lenore can do the ironing for her family's business in h hours. Her daughter would take $h+2$ hours to get the ironing done. If Lenore and her daughter work together, using 2 irons, the number of hours it would take them to do all the ironing is $\dfrac{1}{\frac{1}{h}+\frac{1}{h+2}}$.

ⓐ Simplify the complex fraction $\dfrac{1}{\frac{1}{h}+\frac{1}{h+2}}$.

ⓑ Find the number of hours it would take Lenore and her daughter, working together, to get the ironing done if $h = 4$.

Writing Exercises

301. In this section, you learned to simplify the complex fraction $\dfrac{\frac{3}{x+2}}{\frac{x}{x^2-4}}$ two ways:

rewriting it as a division problem

multiplying the numerator and denominator by the LCD

Which method do you prefer? Why?

302. Efraim wants to start simplifying the complex fraction $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$ by cancelling the variables from the numerator and denominator. Explain what is wrong with Efraim's plan.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
simplify a complex rational expression by writing it as division.			
simplify a complex rational expression by using the LCD.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

8.6 Solve Rational Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve rational equations
- Solve a rational equation for a specific variable

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Solve: $\frac{1}{6}x + \frac{1}{2} = \frac{1}{3}$.
   If you missed this problem, review **Example 2.48**.
2. Solve: $n^2 - 5n - 36 = 0$.
   If you missed this problem, review **Example 7.73**.
3. Solve for y in terms of x: $5x + 2y = 10$ for y.
   If you missed this problem, review **Example 2.65**.

After defining the terms *expression* and *equation* early in **Foundations**, we have used them throughout this book. We have *simplified* many kinds of *expressions* and *solved* many kinds of *equations*. We have simplified many rational expressions so far in this chapter. Now we will solve rational equations.

The definition of a rational equation is similar to the definition of equation we used in **Foundations**.

Rational Equation

A **rational equation** is two rational expressions connected by an equal sign.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	**Rational Equation**
$\frac{1}{8}x + \frac{1}{2}$	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$
$\frac{y+6}{y^2 - 36}$	$\frac{y+6}{y^2 - 36} = y + 1$
$\frac{1}{n-3} + \frac{1}{n+4}$	$\frac{1}{n-3} + \frac{1}{n+4} = \frac{15}{n^2 + n - 12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

Here is an example we did when we worked with linear equations:

	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$	LCD = 8
We multiplied both sides by the LCD.	$8\left(\frac{1}{8}x + \frac{1}{2}\right) = 8\left(\frac{1}{4}\right)$	
Then we distributed.	$8 \cdot \frac{1}{8}x + 8 \cdot \frac{1}{2} = 8 \cdot \frac{1}{4}$	
We simplified—and then we had an equation with no fractions.	$x + 4 = 2$	
Finally, we solved that equation.	$x + 4 - 4 = 2 - 4$	
	$x = -2$	

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then we will have an equation that does not contain rational expressions and thus is much easier for us to solve.

But because the original equation may have a variable in a denominator we must be careful that we don't end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an *extraneous solution*.

Extraneous Solution to a Rational Equation

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, *c*, by writing $x \neq c$ next to the equation.

EXAMPLE 8.59 HOW TO SOLVE EQUATIONS WITH RATIONAL EXPRESSIONS

Solve: $\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$.

Solution

Step 1. Note any value of the variable that would make any denominator zero.	If $x = 0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.	$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}, x \neq 0$
Step 2. Find the least common denominator of *all* denominators in the equation.	Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$.	The LCD is $6x$.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.	Multiply both sides of the equation by the LCD, $6x$.	$6x\cdot\left(\frac{1}{x}+\frac{1}{3}\right)=6x\cdot\left(\frac{5}{6}\right)$
	Use the Distributive Property.	$6x\cdot\frac{1}{x}+6x\cdot\frac{1}{3}=6x\cdot\left(\frac{5}{6}\right)$
	Simplify – and notice, no more fractions!	$6+2x=5x$
Step 4. Solve the resulting equation.	Simplify.	$6=3x$ $2=x$
Step 5. Check. • If any values found in Step 1 are algebraic solutions, discard them. • Check any remaining solutions in the original equation.	We did not get 0 as an algebraic solution.	$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$
	We substitute $x=2$ into the original equation.	$\frac{1}{2}+\frac{1}{3}\stackrel{?}{=}\frac{5}{6}$ $\frac{3}{6}+\frac{2}{6}\stackrel{?}{=}\frac{5}{6}$ $\frac{5}{6}=\frac{5}{6}\checkmark$

TRY IT : : 8.117 Solve: $\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$.

TRY IT : : 8.118 Solve: $\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$.

The steps of this method are shown below.

HOW TO : : SOLVE EQUATIONS WITH RATIONAL EXPRESSIONS.

Step 1. Note any value of the variable that would make any denominator zero.
Step 2. Find the least common denominator of *all* denominators in the equation.
Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
Step 4. Solve the resulting equation.
Step 5. Check.
- If any values found in Step 1 are algebraic solutions, discard them.
- Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

EXAMPLE 8.60

Solve: $1-\frac{5}{y}=-\frac{6}{y^2}$.

Solution

	$1-\frac{5}{y}=-\frac{6}{y^2}$
Note any value of the variable that would make any denominator zero.	$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$
Find the least common denominator of all denominators in the equation. The LCD is y^2.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Distribute.	$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Multiply.	$y^2-5y=-6$
Solve the resulting equation. First write the quadratic equation in standard form.	$y^2-5y+6=0$
Factor.	$(y-2)(y-3)=0$
Use the Zero Product Property.	$y-2=0$ or $y-3=0$
Solve.	$y=2$ or $y=3$
Check.	
We did not get 0 as an algebraic solution.	

Check $y=2$ and $y=3$ in the original equation.

$$\begin{array}{ll}
1-\frac{5}{y}=-\frac{6}{y^2} & 1-\frac{5}{y}=-\frac{6}{y^2} \\
1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{2^2} & 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{3^2} \\
1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} & 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9} \\
\frac{2}{2}-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} & \frac{3}{3}-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9} \\
-\frac{3}{2}\stackrel{?}{=}-\frac{6}{4} & -\frac{2}{3}\stackrel{?}{=}-\frac{6}{9} \\
-\frac{3}{2}=-\frac{3}{2}\checkmark & -\frac{2}{3}=-\frac{2}{3}\checkmark
\end{array}$$

TRY IT : : 8.119 Solve: $1-\frac{2}{a}=\frac{15}{a^2}$.

TRY IT : : 8.120 Solve: $1-\frac{4}{b}=\frac{12}{b^2}$.

EXAMPLE 8.61

Solve: $\frac{5}{3u-2}=\frac{3}{2u}$.

Solution

	$\frac{5}{3u-2} = \frac{3}{2u}$
Note any value of the variable that would make any denominator zero.	$\frac{5}{3u-2} = \frac{3}{2u},\ u \neq \frac{2}{3},\ u \neq 0$
Find the least common denominator of all denominators in the equation. The LCD is $2u(3u-2)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$2u(3u-2)\left(\frac{5}{3u-2}\right) = 2u(3u-2)\left(\frac{3}{2u}\right)$
Remove common factors.	$2u\cancel{(3u-2)}\left(\frac{5}{\cancel{3u-2}}\right) = \cancel{2u}(3u-2)\left(\frac{3}{\cancel{2u}}\right)$
Simplify.	$2u(5) = (3u-2)(3)$
Multiply.	$10u = 9u - 6$
Solve the resulting equation.	$u = -6$
We did not get 0 or $\frac{2}{3}$ as algebraic solutions.	

Check $u = -6$ in the original equation.

$$\frac{5}{3u-2} = \frac{3}{2u}$$

$$\frac{5}{3(-6)-2} \stackrel{?}{=} \frac{3}{2(-6)}$$

$$\frac{5}{-20} \stackrel{?}{=} \frac{3}{-12}$$

$$-\frac{1}{4} = -\frac{1}{4}\checkmark$$

TRY IT : : 8.121 Solve: $\frac{1}{x-1} = \frac{2}{3x}$.

TRY IT : : 8.122 Solve: $\frac{3}{5n+1} = \frac{2}{3n}$.

When one of the denominators is a quadratic, remember to factor it first to find the LCD.

EXAMPLE 8.62

Solve: $\frac{2}{p+2} + \frac{4}{p-2} = \frac{p-1}{p^2-4}$.

Solution

	$\frac{2}{p+2} + \frac{4}{p-2} = \frac{p-1}{p^2-4}$
Note any value of the variable that would make any denominator zero.	$\frac{2}{p+2} + \frac{4}{p-2} = \frac{p-1}{(p+2)(p+2)},\ p \neq -2,\ p \neq 2$

Find the least common denominator of all denominators in the equation. The LCD is $(p+2)(p-2)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(p+2)(p-2)\left(\frac{2}{p+2}+\frac{4}{p-2}\right)=(p+2)(p-2)\left(\frac{p-1}{p^2-4}\right)$
Distribute.	$(p+2)(p-2)\frac{2}{p+2}+(p+2)(p-2)\frac{4}{p-2}=(p+2)(p-2)\left(\frac{p-1}{p^2-4}\right)$
Remove common factors.	$\cancel{(p+2)}(p-2)\frac{2}{\cancel{p+2}}+(p+2)\cancel{(p-2)}\frac{4}{\cancel{p-2}}=\cancel{(p+2)(p-2)}\left(\frac{p-1}{\cancel{p^2-4}}\right)$
Simplify.	$2(p-2)+4(p+2)=p-1$
Distribute.	$2p-4+4p+8=p-1$
Solve.	$6p+4=p-1$
	$5p=-5$
	$p=-1$
We did not get 2 or -2 as algebraic solutions.	

Check $p=-1$ in the original equation.

$$\frac{2}{p+2}+\frac{4}{p-2}=\frac{p-1}{p^2-4}$$

$$\frac{2}{(-1)+2}+\frac{4}{(-1)-2}\stackrel{?}{=}\frac{(-1)-1}{(-1)^2-4}$$

$$\frac{2}{1}+\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$$

$$\frac{6}{3}-\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$$

$$\frac{2}{3}=\frac{2}{3}\checkmark$$

TRY IT : : 8.123 Solve: $\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$.

TRY IT : : 8.124 Solve: $\frac{5}{y+3}+\frac{2}{y-3}=\frac{5}{y^2-9}$.

EXAMPLE 8.63

Solve: $\frac{4}{q-4}-\frac{3}{q-3}=1$.

Solution

	$\frac{4}{q-4} - \frac{3}{q-3} = 1$
Note any value of the variable that would make any denominator zero.	$\frac{4}{q-4} + \frac{3}{q-3} = 1, \quad q \neq 4, q \neq 3$
Find the least common denominator of all denominators in the equation. The LCD is $(q-4)(q-3)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(q-4)(q-3)\left(\frac{4}{q-4} - \frac{3}{q-3}\right) = (q-4)(q-3)(1)$
Distribute.	$(q-4)(q-3)\left(\frac{4}{q-4}\right) - (q-4)(q-3)\left(\frac{3}{q-3}\right) = (q-4)(q-3)(1)$
Remove common factors.	$\cancel{(q-4)}(q-3)\left(\frac{4}{\cancel{q-4}}\right) - (q-4)\cancel{(q-3)}\left(\frac{3}{\cancel{q-3}}\right) = (q-4)(q-3)(1)$
Simplify.	$4(q-3) - 3(q-4) = (q-4)(q-3)$
Simplify.	$4q - 12 - 3q + 12 = q^2 - 7q + 12$
Combine like terms.	$q = q^2 - 7q + 12$
Solve. First write in standard form.	$0 = q^2 - 8q + 12$
Factor.	$0 = (q-2)(q-6)$
Use the Zero Product Property.	$q = 2$ or $q = 6$
We did not get 4 or 3 as algebraic solutions.	

Check $q = 2$ and $q = 6$ in the original equation.

$$\frac{4}{q-4} - \frac{3}{q-3} = 1 \qquad \frac{4}{q-4} - \frac{3}{q-3} = 1$$

$$\frac{4}{2-4} - \frac{3}{2-3} \stackrel{?}{=} 1 \qquad \frac{4}{6-4} - \frac{3}{6-3} \stackrel{?}{=} 1$$

$$\frac{4}{-2} - \frac{3}{-1} \stackrel{?}{=} 1 \qquad \frac{4}{2} - \frac{3}{1} \stackrel{?}{=} 1$$

$$-2 - (-3) \stackrel{?}{=} 1 \qquad 2 - 1 \stackrel{?}{=} 1$$

$$1 = 1 \checkmark \qquad 1 = 1 \checkmark$$

TRY IT : : 8.125 Solve: $\frac{2}{x+5} - \frac{1}{x-1} = 1.$

TRY IT : : 8.126 Solve: $\frac{3}{x+8} - \frac{2}{x-2} = 1.$

EXAMPLE 8.64

Solve: $\frac{m+11}{m^2 - 5m + 4} = \frac{5}{m-4} - \frac{3}{m-1}.$

Solution

	$\frac{m+11}{m^2-5m+4} = \frac{5}{m-4} = \frac{3}{m-1}$
Factor all the denominators, so we can note any value of the variable the would make any denominator zero.	$\frac{m+11}{(m-4)(m-1)} = \frac{5}{m-4} - \frac{3}{m-1},\ m \neq 4, m \neq 1$
Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$.	
Clear the fractions.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right) = (m-4)(m-1)\left(\frac{5}{m-4} - \frac{3}{m-1}\right)$
Distribute.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right) = (m-4)(m-1)\frac{5}{m-4} - (m-4)(m-1)\frac{3}{m-1}$
Remove common factors.	$\cancel{(m-4)}\cancel{(m-1)}\left(\frac{m+11}{\cancel{(m-4)}\cancel{(m-1)}}\right) = \cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}} - (m-4)\cancel{(m-1)}\frac{3}{\cancel{m-1}}$
Simplify.	$m+11 = 5(m-1) - 3(m-4)$
Solve the resulting equation.	$m+11 = 5m-5-3m+12$
	$4 = m$
Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution. There is no solution to this equation.	

TRY IT : : 8.127 Solve: $\frac{x+13}{x^2-7x+10} = \frac{6}{x-5} - \frac{4}{x-2}$.

TRY IT : : 8.128 Solve: $\frac{y-14}{y^2+3y-4} = \frac{2}{y+4} + \frac{7}{y-1}$.

The equation we solved in **Example 8.64** had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. Some equations have no solution.

EXAMPLE 8.65

Solve: $\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n}$.

Solution

	$\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n}$
Note any value of the variable that would make any denominator zero.	$\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n},\ n \neq 0$

Find the least common denominator of all denominators in the equation. The LCD is $12n$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$12n\left(\frac{n}{12}+\frac{n+3}{3n}\right)=12n\left(\frac{1}{n}\right)$
Distribute.	$12n\left(\frac{n}{12}\right)+12n\left(\frac{n+3}{3n}\right)=12n\left(\frac{1}{n}\right)$
Remove common factors.	$\cancel{12}n\left(\frac{n}{\cancel{12}}\right)+4\cdot\cancel{3n}\left(\frac{n+3}{\cancel{3n}}\right)=12\cancel{n}\left(\frac{1}{\cancel{n}}\right)$
Simplify.	$n\cdot n+4(n+3)=12\cdot 1$
Solve the resulting equation.	$n^2+4n+12=12$
	$n^2+4n=0$
	$n(n+4)=0$
	$n=0 \text{ or } n=-4$
Check.	
$n=0$ is an extraneous solution.	
Check $n=-4$ in the original equation. $$\frac{n}{12}+\frac{n+3}{3n}=\frac{1}{n}$$ $$\frac{-4}{12}+\frac{-4+3}{3(-4)}\stackrel{?}{=}\frac{1}{-4}$$ $$-\frac{4}{12}+\frac{1}{12}\stackrel{?}{=}-\frac{1}{4}$$ $$-\frac{3}{12}\stackrel{?}{=}-\frac{1}{4}$$ $$-\frac{1}{4}=-\frac{1}{4}\checkmark$$	

TRY IT : : 8.129 Solve: $\frac{x}{18}+\frac{x+6}{9x}=\frac{2}{3x}$.

TRY IT : : 8.130 Solve: $\frac{y+5}{5y}+\frac{y}{15}=\frac{1}{y}$.

EXAMPLE 8.66

Solve: $\frac{y}{y+6}=\frac{72}{y^2-36}+4$.

Solution

	$\frac{y}{y+6}=\frac{72}{y^2-36}+4$
Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$

Find the least common denominator. The LCD is $(y-6)(y+6)$.	
Clear the fractions.	$(y-6)(y+6)\left(\dfrac{y}{y+6}\right)=(y-6)(y+6)\left(\dfrac{72}{(y-6)(y+6)}+4\right)$
Simplify.	$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$
Simplify.	$y(y-6)=72+4(y^2-36)$
Solve the resulting equation.	$y^2-6y=72+4y^2-144$
	$0=3y^2+6y-72$
	$0=3(y^2+2y-24)$
	$0=3(y+6)(y-4)$
	$y=-6,\ y=4$
Check.	
$y=-6$ is an extraneous solution.	

Check $y=4$ in the original equation.

$$\frac{y}{y+6}=\frac{72}{y^2-36}+4$$

$$\frac{4}{4+6}\stackrel{?}{=}\frac{72}{4^2-36}+4$$

$$\frac{4}{10}\stackrel{?}{=}\frac{72}{-20}+4$$

$$\frac{4}{10}\stackrel{?}{=}-\frac{36}{10}+\frac{40}{10}$$

$$\frac{4}{10}=\frac{4}{10}\checkmark$$

TRY IT : : 8.131 Solve: $\dfrac{x}{x+4}=\dfrac{32}{x^2-16}+5.$

TRY IT : : 8.132 Solve: $\dfrac{y}{y+8}=\dfrac{128}{y^2-64}+9.$

EXAMPLE 8.67

Solve: $\dfrac{x}{2x-2}-\dfrac{2}{3x+3}=\dfrac{5x^2-2x+9}{12x^2-12}.$

Solution

	$\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}$
We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}$
Note any value of the variable that would make any denominator zero.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}, x \neq 1, x \neq -1$
Find the least common denominator.The LCD is $12(x-1)(x+1)$	
Clear the fractions.	$12(x-1)(x+1)\left(\frac{x}{2(x-1)} - \frac{2}{3(x+1)}\right) = 12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$
Simplify.	$6(x+1) \cdot x - 4(x-1) \cdot 2 = 5x^2 - 2x + 9$
Simplify.	$6x(x+1) - 4 \cdot 2(x-1) = 5x^2 - 2x + 9$
Solve the resulting equation.	$6x^2 + 6x - 8x + 8 = 5x^2 - 2x + 9$
	$x^2 - 1 = 0$
	$(x-1)(x+1) = 0$
	$x = 1$ or $x = -1$
Check.	
$x = 1$ and $x = -1$ are extraneous solutions. The equation has no solution.	

TRY IT : : 8.133 Solve: $\frac{y}{5y-10} - \frac{5}{3y+6} = \frac{2y^2-19y+54}{15y^2-60}$.

TRY IT : : 8.134 Solve: $\frac{z}{2z+8} - \frac{3}{4z-8} = \frac{3z^2-16z-6}{8z^2+8z-64}$.

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

We'll start with a formula relating distance, rate, and time. We have used it many times before, but not usually in this form.

EXAMPLE 8.68

Solve: $\frac{D}{T} = R$ for T.

Solution

	$\frac{D}{T} = R$ for T
Note any value of the variable that would make any denominator zero.	$\frac{D}{T} = R, T \neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, T.	$T\left(\frac{D}{T}\right) = T(R)$
Simplify.	$D = T \cdot R$
Divide both sides by R to isolate T.	$\frac{D}{R} = \frac{RT}{R}$
Simplify.	$\frac{D}{R} = T$

TRY IT : : 8.135 Solve: $\frac{A}{L} = W$ for L.

TRY IT : : 8.136 Solve: $\frac{F}{A} = M$ for A.

Example 8.69 uses the formula for slope that we used to get the point-slope form of an equation of a line.

EXAMPLE 8.69

Solve: $m = \frac{x-2}{y-3}$ for y.

Solution

	$m = \frac{x-2}{y-3}$ for y
Note any value of the variable that would make any denominator zero.	$m = \frac{x-2}{y-3},\ y \neq 3$
Clear the fractions by multiplying both sides of the equations by the LCD, $y-3$.	$(y-3)m = (y-3)\left(\frac{x-2}{y-3}\right)$
Simplify.	$ym - 3m = x - 2$
Isolate the term with y.	$ym = x - 2 + 3m$
Divide both sides by m to isolate y.	$\frac{ym}{m} = \frac{x-2+3m}{m}$
Simplify.	$y = \frac{x-2+3m}{m}$

TRY IT : : 8.137 Solve: $\frac{y-2}{x+1} = \frac{2}{3}$ for x.

TRY IT : : 8.138 Solve: $x = \frac{y}{1-y}$ for y.

Be sure to follow all the steps in Example 8.70. It may look like a very simple formula, but we cannot solve it instantly for either denominator.

EXAMPLE 8.70

Solve $\frac{1}{c} + \frac{1}{m} = 1$ for c.

Solution

	$\frac{1}{c} + \frac{1}{m} = 1$ for c
Note any value of the variable that would make any denominator zero.	$\frac{1}{c} + \frac{1}{m} = 1,\ c \neq 0,\ m \neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, cm.	$cm\left(\frac{1}{c} + \frac{1}{m}\right) = cm(1)$
Distribute.	$cm\left(\frac{1}{c}\right) + cm\,\frac{1}{m} = cm(1)$
Simplify.	$m + c = cm$
Collect the terms with c to the right.	$m = cm - c$
Factor the expression on the right.	$m = c(m-1)$
To isolate c, divide both sides by $m-1$.	$\frac{m}{m-1} = \frac{c(m-1)}{m-1}$
Simplify by removing common factors.	$\frac{m}{m-1} = c$

Notice that even though we excluded $c = 0$ and $m = 0$ from the original equation, we must also now state that $m \neq 1$.

TRY IT : : 8.139 Solve: $\frac{1}{a} + \frac{1}{b} = c$ for a.

TRY IT : : 8.140 Solve: $\frac{2}{x} + \frac{1}{3} = \frac{1}{y}$ for y.

8.6 EXERCISES

Practice Makes Perfect

Solve Rational Equations

In the following exercises, solve.

303. $\frac{1}{a}+\frac{2}{5}=\frac{1}{2}$

304. $\frac{5}{6}+\frac{3}{b}=\frac{1}{3}$

305. $\frac{5}{2}-\frac{1}{c}=\frac{3}{4}$

306. $\frac{6}{3}-\frac{2}{d}=\frac{4}{9}$

307. $\frac{4}{5}+\frac{1}{4}=\frac{2}{v}$

308. $\frac{3}{7}+\frac{2}{3}=\frac{1}{w}$

309. $\frac{7}{9}+\frac{1}{x}=\frac{2}{3}$

310. $\frac{3}{8}+\frac{2}{y}=\frac{1}{4}$

311. $1-\frac{2}{m}=\frac{8}{m^2}$

312. $1+\frac{4}{n}=\frac{21}{n^2}$

313. $1+\frac{9}{p}=\frac{-20}{p^2}$

314. $1-\frac{7}{q}=\frac{-6}{q^2}$

315. $\frac{1}{r+3}=\frac{4}{2r}$

316. $\frac{3}{t-6}=\frac{1}{t}$

317. $\frac{5}{3v-2}=\frac{7}{4v}$

318. $\frac{8}{2w+1}=\frac{3}{w}$

319. $\frac{3}{x+4}+\frac{7}{x-4}=\frac{8}{x^2-16}$

320. $\frac{5}{y-9}+\frac{1}{y+9}=\frac{18}{y^2-81}$

321. $\frac{8}{z-10}+\frac{7}{z+10}=\frac{5}{z^2-100}$

322. $\frac{9}{a+11}+\frac{6}{a-11}=\frac{7}{a^2-121}$

323. $\frac{1}{q+4}-\frac{7}{q-2}=1$

324. $\frac{3}{r+10}-\frac{4}{r-4}=1$

325. $\frac{1}{t+7}-\frac{5}{t-5}=1$

326. $\frac{2}{s+7}-\frac{3}{s-3}=1$

327. $\frac{v-10}{v^2-5v+4}=\frac{3}{v-1}-\frac{6}{v-4}$

328. $\frac{w+8}{w^2-11w+28}=\frac{5}{w-7}+\frac{2}{w-4}$

329. $\frac{x-10}{x^2+8x+12}=\frac{3}{x+2}+\frac{4}{x+6}$

330. $\frac{y-3}{y^2-4y-5}=\frac{1}{y+1}+\frac{8}{y-5}$

331. $\frac{z}{16}+\frac{z+2}{4z}=\frac{1}{2z}$

332. $\frac{a}{9}+\frac{a+3}{3a}=\frac{1}{a}$

333. $\frac{b+3}{3b}+\frac{b}{24}=\frac{1}{b}$

334. $\frac{c+3}{12c}+\frac{c}{36}=\frac{1}{4c}$

335. $\frac{d}{d+3}=\frac{18}{d^2-9}+4$

336. $\frac{m}{m+5}=\frac{50}{m^2-25}+6$

337. $\frac{n}{n+2}=\frac{8}{m^2-4}+3$

338. $\frac{p}{p+7}=\frac{98}{p^2-49}+8$

339. $\frac{q}{3q-9}-\frac{3}{4q+12}=\frac{7q^2+6q+63}{24q^2-216}$

340. $\frac{r}{3r-15}-\frac{1}{4r+20}=\frac{3r^2+17r+40}{12r^2-300}$

341. $\frac{s}{2s+6}-\frac{2}{5s+5}=\frac{5s^2-3s-7}{10s^2+40s+30}$

342. $\frac{t}{6t-12} - \frac{5}{2t+10}$
$= \frac{t^2 - 23t + 70}{12t^2 + 36t - 120}$

Solve a Rational Equation for a Specific Variable

In the following exercises, solve.

343. $\frac{C}{r} = 2\pi$ for r

344. $\frac{I}{r} = P$ for r

345. $\frac{V}{h} = lw$ for h

346. $\frac{2A}{b} = h$ for b

347. $\frac{v+3}{w-1} = \frac{1}{2}$ for w

348. $\frac{x+5}{2-y} = \frac{4}{3}$ for y

349. $a = \frac{b+3}{c-2}$ for c

350. $m = \frac{n}{2-n}$ for n

351. $\frac{1}{p} + \frac{2}{q} = 4$ for p

352. $\frac{3}{s} + \frac{1}{t} = 2$ for s

353. $\frac{2}{v} + \frac{1}{5} = \frac{1}{2}$ for v

354. $\frac{6}{x} + \frac{2}{3} = \frac{1}{y}$ for y

355. $\frac{m+3}{n-2} = \frac{4}{5}$ for n

356. $\frac{E}{c} = m^2$ for c

357. $\frac{3}{x} - \frac{5}{y} = \frac{1}{4}$ for y

358. $\frac{R}{T} = W$ for T

359. $r = \frac{s}{3-t}$ for t

360. $c = \frac{2}{a} + \frac{b}{5}$ for a

Everyday Math

361. House Painting Alain can paint a house in 4 days. Spiro would take 7 days to paint the same house. Solve the equation $\frac{1}{4} + \frac{1}{7} = \frac{1}{t}$ for t to find the number of days it would take them to paint the house if they worked together.

362. Boating Ari can drive his boat 18 miles with the current in the same amount of time it takes to drive 10 miles against the current. If the speed of the boat is 7 knots, solve the equation $\frac{18}{7+c} = \frac{10}{7-c}$ for c to find the speed of the current.

Writing Exercises

363. Why is there no solution to the equation $\frac{3}{x-2} = \frac{5}{x-2}$?

364. Pete thinks the equation $\frac{y}{y+6} = \frac{72}{y^2-36} + 4$ has two solutions, $y = -6$ and $y = 4$. Explain why Pete is wrong.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve rational equations.			
solve rational equations for a specific variable.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

8.7 Solve Proportion and Similar Figure Applications

Learning Objectives

By the end of this section, you will be able to:

- Solve proportions
- Solve similar figure applications

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Solve $\frac{n}{3} = 30$.

   If you missed this problem, review Example 2.21.
2. The perimeter of a triangular window is 23 feet. The lengths of two sides are ten feet and six feet. How long is the third side?
   If you missed this problem, review Example 3.35.

Solve Proportions

When two rational expressions are equal, the equation relating them is called a *proportion*.

Proportion

A **proportion** is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$.

The proportion is read "a is to b, as c is to d."

The equation $\frac{1}{2} = \frac{4}{8}$ is a proportion because the two fractions are equal. The proportion $\frac{1}{2} = \frac{4}{8}$ is read "1 is to 2 as 4 is to 8."

Proportions are used in many applications to 'scale up' quantities. We'll start with a very simple example so you can see how proportions work. Even if you can figure out the answer to the example right away, make sure you also learn to solve it using proportions.

Suppose a school principal wants to have 1 teacher for 20 students. She could use proportions to find the number of teachers for 60 students. We let *x* be the number of teachers for 60 students and then set up the proportion:

$$\frac{1 \text{ teacher}}{20 \text{ students}} = \frac{x \text{ teachers}}{60 \text{ students}}$$

We are careful to match the units of the numerators and the units of the denominators—teachers in the numerators, students in the denominators.

Since a proportion is an equation with rational expressions, we will solve proportions the same way we solved equations in Solve Rational Equations. We'll multiply both sides of the equation by the LCD to clear the fractions and then solve the resulting equation.

So let's finish solving the principal's problem now. We will omit writing the units until the last step.

$$\frac{1}{20} = \frac{x}{60}$$

Multiply both sides by the LCD, 60.	$\frac{1}{20} \cdot 60 = \frac{x}{60} \cdot 60$
Simplify.	$3 = x$
	The principal needs 3 teachers for 60 students.

Now we'll do a few examples of solving numerical proportions without any units. Then we will solve applications using proportions.

EXAMPLE 8.71

Solve the proportion: $\frac{x}{63} = \frac{4}{7}$.

Solution

	$\frac{x}{63} = \frac{4}{7}$
To isolate x, multiply both sides by the LCD, 63.	$63\left(\frac{x}{63}\right) = 63\left(\frac{4}{7}\right)$
Simplify.	$x = \frac{9 \cdot \cancel{7} \cdot 4}{\cancel{7}}$
Divide the common factors.	$x = 36$
Check. To check our answer, we substitute into the original proportion.	
	$\frac{x}{63} = \frac{4}{7}$
Substitute $x = 36$.	$\frac{36}{63} \stackrel{?}{=} \frac{4}{7}$
Show common factors.	$\frac{4 \cdot 9}{7 \cdot 9} \stackrel{?}{=} \frac{4}{7}$
Simplify.	$\frac{4}{7} = \frac{4}{7}$ ✓

TRY IT : : 8.141 Solve the proportion: $\frac{n}{84} = \frac{11}{12}$.

TRY IT : : 8.142 Solve the proportion: $\frac{y}{96} = \frac{13}{12}$.

When we work with proportions, we exclude values that would make either denominator zero, just like we do for all rational expressions. What value(s) should be excluded for the proportion in the next example?

EXAMPLE 8.72

Solve the proportion: $\frac{144}{a} = \frac{9}{4}$.

Solution

	$\frac{144}{a} = \frac{9}{4}$
Multiply both sides by the LCD.	$\frac{144}{a} \cdot 4a = \frac{9}{4} \cdot 4a$
Remove common factors on each side.	$4 \cdot 144 = a \cdot 9$
Simplify.	$576 = 9a$
Divide both sides by 9.	$\frac{576}{9} = \frac{9a}{9}$

Simplify.	$64 = a$
Check.	
	$\frac{144}{a} = \frac{9}{4}$
Substitute $a = 64$.	$\frac{144}{64} \stackrel{?}{=} \frac{9}{4}$
Show common factors.	$\frac{9 \cdot 16}{4 \cdot 16} \stackrel{?}{=} \frac{9}{4}$
Simplify.	$\frac{9}{4} = \frac{9}{4}$ ✓

TRY IT : : 8.143 Solve the proportion: $\frac{91}{b} = \frac{7}{5}$.

TRY IT : : 8.144 Solve the proportion: $\frac{39}{c} = \frac{13}{8}$.

EXAMPLE 8.73

Solve the proportion: $\frac{n}{n+14} = \frac{5}{7}$.

Solution

	$\frac{n}{n+14} = \frac{5}{7}$
Multiply both sides by the LCD.	$7(n+14)\left(\frac{n}{n+14}\right) = 7(n+14)\left(\frac{5}{7}\right)$
Remove common factors on each side.	$7n = 5(n+14)$
Simplify.	$7n = 5n + 70$
Solve for n.	$2n = 70$
	$n = 35$
Check.	
	$\frac{n}{n+14} = \frac{5}{7}$
Substitute $n = 35$.	$\frac{35}{35+14} \stackrel{?}{=} \frac{5}{7}$
Simplify.	$\frac{35}{49} \stackrel{?}{=} \frac{5}{7}$
Show common factors.	$\frac{5 \cdot 7}{7 \cdot 7} \stackrel{?}{=} \frac{5}{7}$
Simplify.	$\frac{5}{7} = \frac{5}{7}$ ✓

TRY IT : : 8.145 Solve the proportion: $\frac{y}{y+55} = \frac{3}{8}$.

TRY IT : : 8.146 Solve the proportion: $\frac{z}{z-84} = -\frac{1}{5}$.

EXAMPLE 8.74

Solve: $\frac{p+12}{9} = \frac{p-12}{6}$.

Solution

	$\frac{p+12}{9} = \frac{p-12}{6}$
Multiply both sides by the LCD, 18.	$18\left(\frac{p+12}{9}\right) = 18\left(\frac{p-12}{6}\right)$
Simplify.	$2(p+12) = 3(p-12)$
Distribute.	$2p + 24 = 3p - 36$
Solve for p.	$60 = p$
Check.	
	$\frac{p+12}{9} = \frac{p-12}{6}$
Substitute $p = 60$.	$\frac{60+12}{9} \stackrel{?}{=} \frac{60-12}{6}$
Simplify.	$\frac{72}{9} \stackrel{?}{=} \frac{48}{6}$
Divide.	$8 = 8\checkmark$

TRY IT : : 8.147 Solve: $\frac{v+30}{8} = \frac{v+66}{12}$.

TRY IT : : 8.148 Solve: $\frac{2x+15}{9} = \frac{7x+3}{15}$.

To solve applications with proportions, we will follow our usual strategy for solving applications. But when we set up the proportion, we must make sure to have the units correct—the units in the numerators must match and the units in the denominators must match.

EXAMPLE 8.75

When pediatricians prescribe acetaminophen to children, they prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of the child's weight. If Zoe weighs 80 pounds, how many milliliters of acetaminophen will her doctor prescribe?

Solution

Identify what we are asked to find, and choose a variable to represent it.	How many ml of acetaminophen will the doctor prescribe?
	Let $a = \text{ml}$ of acetaminophen.
Write a sentence that gives the information to find it.	If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?

Translate into a proportion–be careful of the units. $\frac{\text{ml}}{\text{pounds}} = \frac{\text{ml}}{\text{pounds}}$	$\frac{5}{25} = \frac{a}{80}$
Multiply both sides by the LCD, 400.	$400\left(\frac{5}{25}\right) = 400\left(\frac{a}{80}\right)$
Remove common factors on each side.	$\cancel{25} \cdot 16\left(\frac{5}{\cancel{25}}\right) = \cancel{80} \cdot 5\left(\frac{a}{\cancel{80}}\right)$
Simplify, but don't multiply on the left. Notice what the next step will be.	$16 \cdot 5 = 5a$
Solve for a.	$\frac{16 \cdot 5}{5} = \frac{5a}{5}$
Check.	$16 = a$
Is the answer reasonable?	
Yes, since 80 is about 3 times 25, the medicine should be about 3 times 5. So 16 ml makes sense.	
Substitute $a = 16$ in the original proportion. $\frac{5}{25} = \frac{a}{80}$ $\frac{5}{25} \stackrel{?}{=} \frac{16}{80}$ $\frac{1}{5} = \frac{1}{5}$ ✓	
Write a complete sentence.	The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

TRY IT : : 8.149

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child’s weight. How many milliliters of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

TRY IT : : 8.150

For every 1 kilogram (kg) of a child’s weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

EXAMPLE 8.76

A 16-ounce iced caramel macchiato has 230 calories. How many calories are there in a 24-ounce iced caramel macchiato?

Solution

Identify what we are asked to find, and choose a variable to represent it.	How many calories are in a 24 ounce iced caramel macchiato?
	Let $c = \text{calories}$ in 24 ounces.
Write a sentence that gives the information to find it.	If there are 230 calories in 16 ounces, then how many calories are in 24 ounces?

Translate into a proportion–be careful of the units. $\frac{\text{calories}}{\text{ounce}} = \frac{\text{calories}}{\text{ounce}}$	$\frac{230}{16} = \frac{c}{24}$
Multiply both sides by the LCD, 48.	$48\left(\frac{230}{16}\right) = 48\left(\frac{c}{24}\right)$
Remove common factors on each side.	$\cancel{16} \cdot 3\left(\frac{230}{\cancel{16}}\right) = \cancel{24} \cdot 2\left(\frac{c}{\cancel{24}}\right)$
Simplify.	$690 = 2c$
Solve for c.	$\frac{690}{2} = \frac{2c}{2}$
	$345 = c$
Check.	
Is the answer reasonable?	
Yes, 345 calories for 24 ounces is more than 290 calories for 16 ounces, but not too much more.	
Substitute $c = 345$ in the original proportion. $\frac{230}{16} = \frac{c}{24}$ $\frac{230}{16} \stackrel{?}{=} \frac{345}{24}$ $\frac{115}{8} = \frac{115}{8}$ ✓	
Write a complete sentence.	There are 345 calories in a 24-ounce iced caramel macchiato.

TRY IT : : 8.151

At a fast-food restaurant, a 22-ounce chocolate shake has 850 calories. How many calories are in their 12-ounce chocolate shake? Round your answer to nearest whole number.

TRY IT : : 8.152

Yaneli loves Starburst candies, but wants to keep her snacks to 100 calories. If the candies have 160 calories for 8 pieces, how many pieces can she have in her snack?

EXAMPLE 8.77

Josiah went to Mexico for spring break and changed $325 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 12.54 Mexican pesos. How many Mexican pesos did he get for his trip?

Solution

What are you asked to find?	How many Mexican pesos did Josiah get?
Assign a variable.	Let $p =$ the number of Mexican pesos.

Write a sentence that gives the information to find it.	If \$1 US is equal to 12.54 Mexican pesos, then \$325 is how many pesos?
Translate into a proportion–be careful of the units.	
$\frac{\$}{\text{pesos}} = \frac{\$}{\text{pesos}}$	$\frac{1}{12.54} = \frac{325}{p}$
Multiply both sides by the LCD, $12.54p$.	$12.54p\left(\frac{1}{12.54}\right) = 12.54p\left(\frac{325}{p}\right)$
Remove common factors on each side.	$\cancel{12.54}p\left(\frac{1}{\cancel{12.54}}\right) = 12.54\cancel{p}\left(\frac{325}{\cancel{p}}\right)$
Simplify.	$p = 4075.5$
Check.	
Is the answer reasonable?	
Yes, \$100 would be 1,254 pesos. \$325 is a little more than 3 times this amount, so our answer of 4075.5 pesos makes sense.	
Substitute $p = 4075.5$ in the original proportion. Use a calculator. $\frac{1}{12.54} = \frac{325}{p}$ $\frac{1}{12.54} \stackrel{?}{=} \frac{325}{4075.5}$ $0.07874... = 0.07874...$ ✓	
Write a complete sentence.	Josiah got 4075.5 pesos for his spring break trip.

TRY IT : : 8.153

Yurianna is going to Europe and wants to change \$800 dollars into Euros. At the current exchange rate, \$1 US is equal to 0.738 Euro. How many Euros will she have for her trip?

TRY IT : : 8.154

Corey and Nicole are traveling to Japan and need to exchange \$600 into Japanese yen. If each dollar is 94.1 yen, how many yen will they get?

In the example above, we related the number of pesos to the number of dollars by using a proportion. We could say the number of pesos *is proportional to* the number of dollars. If two quantities are related by a proportion, we say that they are proportional.

Solve Similar Figure Applications

When you shrink or enlarge a photo on a phone or tablet, figure out a distance on a map, or use a pattern to build a bookcase or sew a dress, you are working with **similar figures**. If two figures have exactly the same shape, but different sizes, they are said to be *similar*. One is a scale model of the other. All their corresponding angles have the same measures and their corresponding sides are in the same ratio.

Similar Figures

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides are in the same ratio.

For example, the two triangles in **Figure 8.2** are similar. Each side of ΔABC is 4 times the length of the corresponding side of ΔXYZ.

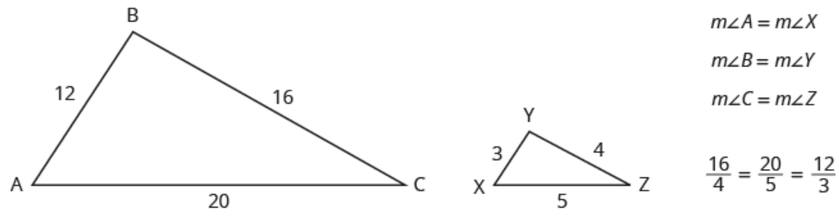
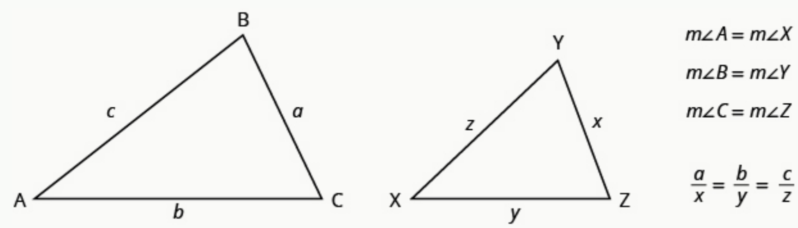


Figure 8.2

This is summed up in the Property of Similar Triangles.

Property of Similar Triangles

If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides are in the same ratio.



To solve applications with similar figures we will follow the Problem-Solving Strategy for Geometry Applications we used earlier.

HOW TO : : SOLVE GEOMETRY APPLICATIONS.

Step 1. **Read** the problem and make all the words and ideas are understood. Draw the figure and label it with the given information.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for by choosing a variable to represent it.

Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

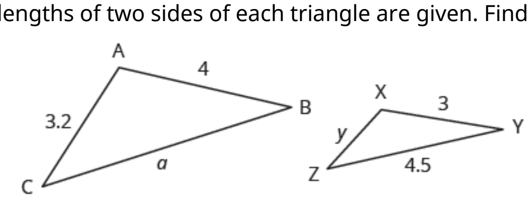
Step 5. **Solve the equation** using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

EXAMPLE 8.78

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given. Find the lengths of the third sides.

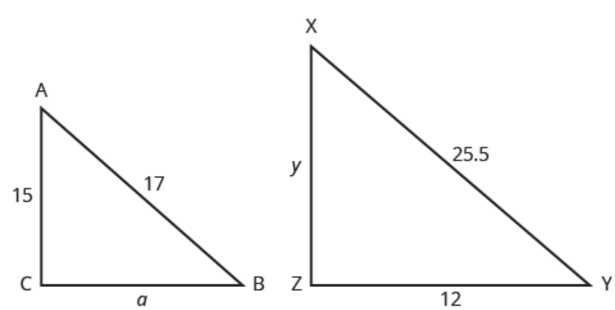


Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	Figure is given.
Step 2. Identify what we are looking for.	the length of the sides of similar triangles
Step 3. Name the variables.	Let $a =$ length of the third side of ΔABC. $y =$ length of the third side of ΔXYZ
Step 4. Translate.	Since the triangles are similar, the corresponding sides are proportional.
We need to write an equation that compares the side we are looking for to a known ratio. Since the side *AB* = 4 corresponds to the side *XY* = 3 we know $\frac{AB}{XY} = \frac{4}{3}$. So we write equations with $\frac{AB}{XY}$ to find the sides we are looking for. Be careful to match up corresponding sides correctly.	$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$. To find *a*: sides of large triangle / sides of small triangle $\frac{AB}{XY} = \frac{BC}{YZ}$ To find *y*: $\frac{AB}{XY} = \frac{AC}{XZ}$
Substitute.	$\frac{4}{3} = \frac{a}{4.5}$ $\frac{4}{3} = \frac{3.2}{y}$
Step 5. Solve the equation.	$3a = 4(4.5)$ $4y = 3(3.2)$
	$a = 6$ $y = 2.4$
Step 6. Check.	
$\frac{4}{3} \stackrel{?}{=} \frac{6}{4.5}$ $4(4.5) \stackrel{?}{=} 6(3)$ $18 = 18$ ✓	$\frac{4}{3} \stackrel{?}{=} \frac{3.2}{2.4}$ $4(2.4) \stackrel{?}{=} 3.2(3)$ $9.6 = 9.6$ ✓
Step 7. Answer the question.	The third side of ΔABC is 6 and the third side of ΔXYZ is 2.4.

TRY IT : : 8.155

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure.



Find the length of side a.

TRY IT : : 8.156

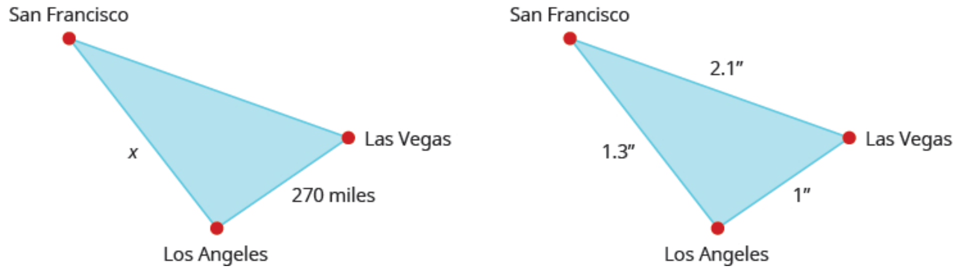
ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure.

Find the length of side y.

The next example shows how similar triangles are used with maps.

EXAMPLE 8.79

On a map, San Francisco, Las Vegas, and Los Angeles form a triangle whose sides are shown in the figure below. If the actual distance from Los Angeles to Las Vegas is 270 miles find the distance from Los Angeles to San Francisco.

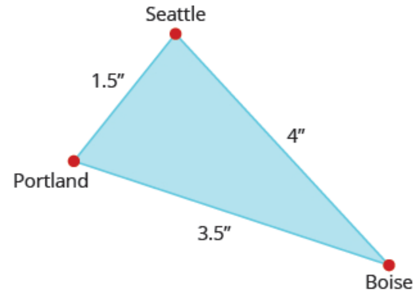


Solution

Read the problem. Draw the figures and label with the given information.	The figures are shown above.
Identify what we are looking for.	The actual distance from Los Angeles to San Francisco.
Name the variables.	Let $x =$ distance from Los Angeles to San Francisco.
Translate into an equation. Since the triangles are similar, the corresponding sides are proportional. We'll make the numerators "miles" and the denominators "inches."	$\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$
Solve the equation.	$1.3\left(\frac{x}{1.3}\right) = 1.3\left(\frac{270}{1}\right)$
	$x = 351$
Check.	
On the map, the distance from Los Angeles to San Francisco is more than the distance from Los Angeles to Las Vegas. Since 351 is more than 270 the answer makes sense.	
Check $x = 351$ in the original proportion. Use a calculator. $\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$ $\frac{351 \text{ miles}}{1.3 \text{ inches}} \overset{?}{=} \frac{270 \text{ miles}}{1 \text{ inch}}$ $\frac{270 \text{ miles}}{1 \text{ inch}} = \frac{270 \text{ miles}}{1 \text{ inch}}$ ✓	
Answer the question.	The distance from Los Angeles to San Francisco is 351 miles.

TRY IT : : 8.157

On the map, Seattle, Portland, and Boise form a triangle whose sides are shown in the figure below. If the actual distance from Seattle to Boise is 400 miles, find the distance from Seattle to Portland.



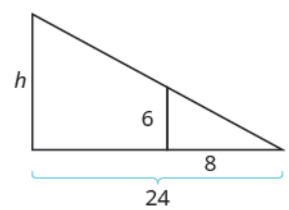
TRY IT : : 8.158 Using the map above, find the distance from Portland to Boise.

We can use similar figures to find heights that we cannot directly measure.

EXAMPLE 8.80

Tyler is 6 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a tree was 24 feet long. Find the height of the tree.

Solution

Read the problem and draw a figure.	
We are looking for h, the height of the tree.	
We will use similar triangles to write an equation.	
The small triangle is similar to the large triangle.	$\frac{h}{24} = \frac{6}{8}$
Solve the proportion.	$24\left(\frac{6}{8}\right) = 24\left(\frac{h}{24}\right)$
Simplify.	$18 = h$
Check.	
Tyler's height is less than his shadow's length so it makes sense that the tree's height is less than the length of its shadow.	

Check $h = 18$ in the original proportion.

$$\frac{6}{8} = \frac{h}{24}$$

$$\frac{6}{8} \stackrel{?}{=} \frac{18}{24}$$

$$\frac{3}{4} = \frac{3}{4} \checkmark$$

TRY IT : : 8.159

A telephone pole casts a shadow that is 50 feet long. Nearby, an 8 foot tall traffic sign casts a shadow that is 10 feet long. How tall is the telephone pole?

TRY IT : : 8.160

A pine tree casts a shadow of 80 feet next to a 30-foot tall building which casts a 40 feet shadow. How tall is the pine tree?

8.7 EXERCISES

Practice Makes Perfect

Solve Proportions

In the following exercises, solve.

365. $\frac{x}{56} = \frac{7}{8}$

366. $\frac{n}{91} = \frac{8}{13}$

367. $\frac{49}{63} = \frac{z}{9}$

368. $\frac{56}{72} = \frac{y}{9}$

369. $\frac{5}{a} = \frac{65}{117}$

370. $\frac{4}{b} = \frac{64}{144}$

371. $\frac{98}{154} = \frac{-7}{p}$

372. $\frac{72}{156} = \frac{-6}{q}$

373. $\frac{a}{-8} = \frac{-42}{48}$

374. $\frac{b}{-7} = \frac{-30}{42}$

375. $\frac{2.7}{j} = \frac{0.9}{0.2}$

376. $\frac{2.8}{k} = \frac{2.1}{1.5}$

377. $\frac{a}{a+12} = \frac{4}{7}$

378. $\frac{b}{b-16} = \frac{11}{9}$

379. $\frac{c}{c-104} = -\frac{5}{8}$

380. $\frac{d}{d-48} = -\frac{13}{3}$

381. $\frac{m+90}{25} = \frac{m+30}{15}$

382. $\frac{n+10}{4} = \frac{40-n}{6}$

383. $\frac{2p+4}{8} = \frac{p+18}{6}$

384. $\frac{q-2}{2} = \frac{2q-7}{18}$

385. Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?

386. Brianna, who weighs 6 kg, just received her shots and needs a pain killer. The pain killer is prescribed for children at 15 milligrams (mg) for every 1 kilogram (kg) of the child's weight. How many milligrams will the doctor prescribe?

387. A veterinarian prescribed Sunny, a 65 pound dog, an antibacterial medicine in case an infection emerges after her teeth were cleaned. If the dosage is 5 mg for every pound, how much medicine was Sunny given?

388. Belle, a 13 pound cat, is suffering from joint pain. How much medicine should the veterinarian prescribe if the dosage is 1.8 mg per pound?

389. A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12 ounces of the drink?

390. One 12 ounce can of soda has 150 calories. If Josiah drinks the big 32 ounce size from the local mini-mart, how many calories does he get?

391. A new 7 ounce lemon ice drink is advertised for having only 140 calories. How many ounces could Sally drink if she wanted to drink just 100 calories?

392. Reese loves to drink healthy green smoothies. A 16 ounce serving of smoothie has 170 calories. Reese drinks 24 ounces of these smoothies in one day. How many calories of smoothie is he consuming in one day?

393. Janice is traveling to Canada and will change $250 US dollars into Canadian dollars. At the current exchange rate, $1 US is equal to $1.01 Canadian. How many Canadian dollars will she get for her trip?

394. Todd is traveling to Mexico and needs to exchange $450 into Mexican pesos. If each dollar is worth 12.29 pesos, how many pesos will he get for his trip?

395. Steve changed $600 into 480 Euros. How many Euros did he receive for each US dollar?

396. Martha changed $350 US into 385 Australian dollars. How many Australian dollars did she receive for each US dollar?

397. When traveling to Great Britain, Bethany exchanged her $900 into 570 British pounds. How many pounds did she receive for each American dollar?

398. A missionary commissioned to South Africa had to exchange his \$500 for the South African Rand which is worth 12.63 for every dollar. How many Rand did he have after the exchange?

399. Ronald needs a morning breakfast drink that will give him at least 390 calories. Orange juice has 130 calories in one cup. How many cups does he need to drink to reach his calorie goal?

400. Sarah drinks a 32-ounce energy drink containing 80 calories per 12 ounce. How many calories did she drink?

401. Elizabeth is returning to the United States from Canada. She changes the remaining 300 Canadian dollars she has to \$230.05 in American dollars. What was \$1 worth in Canadian dollars?

402. Ben needs to convert \$1000 to the Japanese Yen. One American dollar is worth 123.3 Yen. How much Yen will he have?

403. A golden retriever weighing 85 pounds has diarrhea. His medicine is prescribed as 1 teaspoon per 5 pounds. How much medicine should he be given?

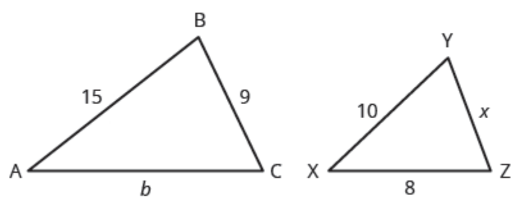
404. Five-year-old Lacy was stung by a bee. The dosage for the anti-itch liquid is 150 mg for her weight of 40 pounds. What is the dosage per pound?

405. Karen eats $\frac{1}{2}$ cup of oatmeal that counts for 2 points on her weight loss program. Her husband, Joe, can have 3 points of oatmeal for breakfast. How much oatmeal can he have?

406. An oatmeal cookie recipe calls for $\frac{1}{2}$ cup of butter to make 4 dozen cookies. Hilda needs to make 10 dozen cookies for the bake sale. How many cups of butter will she need?

Solve Similar Figure Applications

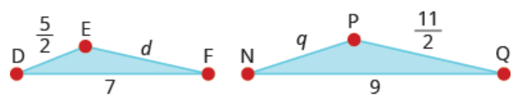
In the following exercises, ΔABC is similar to ΔXYZ. Find the length of the indicated side.



407. side *b*

408. side *x*

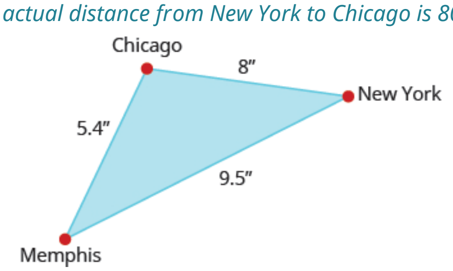
In the following exercises, ΔDEF is similar to ΔNPQ.



409. Find the length of side *d*.

410. Find the length of side *q*.

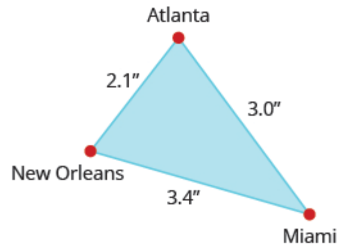
In the following two exercises, use the map shown. On the map, New York City, Chicago, and Memphis form a triangle whose sides are shown in the figure below. The actual distance from New York to Chicago is 800 miles.



411. Find the actual distance from New York to Memphis.

412. Find the actual distance from Chicago to Memphis.

In the following two exercises, use the map shown. On the map, Atlanta, Miami, and New Orleans form a triangle whose sides are shown in the figure below. The actual distance from Atlanta to New Orleans is 420 miles.



413. Find the actual distance from New Orleans to Miami.

414. Find the actual distance from Atlanta to Miami.

415. A 2 foot tall dog casts a 3 foot shadow at the same time a cat casts a one foot shadow. How tall is the cat?

416. Larry and Tom were standing next to each other in the backyard when Tom challenged Larry to guess how tall he was. Larry knew his own height is 6.5 feet and when they measured their shadows, Larry's shadow was 8 feet and Tom's was 7.75 feet long. What is Tom's height?

417. The tower portion of a windmill is 212 feet tall. A six foot tall person standing next to the tower casts a seven foot shadow. How long is the windmill's shadow?

418. The height of the Statue of Liberty is 305 feet. Nicole, who is standing next to the statue, casts a 6 foot shadow and she is 5 feet tall. How long should the shadow of the statue be?

Everyday Math

419. Heart Rate At the gym, Carol takes her pulse for 10 seconds and counts 19 beats.

ⓐ How many beats per minute is this?

ⓑ Has Carol met her target heart rate of 140 beats per minute?

420. Heart Rate Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds.

ⓐ How many beats per minute is this?

ⓑ Has Kevin met his target heart rate?

421. Cost of a Road Trip Jesse's car gets 30 miles per gallon of gas.

ⓐ If Las Vegas is 285 miles away, how many gallons of gas are needed to get there and then home?

ⓑ If gas is $3.09 per gallon, what is the total cost of the gas for the trip?

422. Cost of a Road Trip Danny wants to drive to Phoenix to see his grandfather. Phoenix is 370 miles from Danny's home and his car gets 18.5 miles per gallon.

ⓐ How many gallons of gas will Danny need to get to and from Phoenix?

ⓑ If gas is $3.19 per gallon, what is the total cost for the gas to drive to see his grandfather?

423. Lawn Fertilizer Phil wants to fertilize his lawn. Each bag of fertilizer covers about 4,000 square feet of lawn. Phil's lawn is approximately 13,500 square feet. How many bags of fertilizer will he have to buy?

424. House Paint April wants to paint the exterior of her house. One gallon of paint covers about 350 square feet, and the exterior of the house measures approximately 2000 square feet. How many gallons of paint will she have to buy?

425. Cooking Natalia's pasta recipe calls for 2 pounds of pasta for 1 quart of sauce. How many pounds of pasta should Natalia cook if she has 2.5 quarts of sauce?

426. Heating Oil A 275 gallon oil tank costs $400 to fill. How much would it cost to fill a 180 gallon oil tank?

Writing Exercises

427. Marisol solves the proportion $\frac{144}{a} = \frac{9}{4}$ by 'cross multiplying', so her first step looks like $4 \cdot 144 = 9 \cdot a$. Explain how this differs from the method of solution shown in **Example 8.72**.

428. Find a printed map and then write and solve an application problem similar to **Example 8.79**.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve proportions.			
solve similar figure applications.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

8.8 Solve Uniform Motion and Work Applications

Learning Objectives

By the end of this section, you will be able to:

- Solve uniform motion applications
- Solve work applications

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. An express train and a local bus leave Chicago to travel to Champaign. The express bus can make the trip in 2 hours and the local bus takes 5 hours for the trip. The speed of the express bus is 42 miles per hour faster than the speed of the local bus. Find the speed of the local bus.
If you missed this problem, review **Example 3.48**.
2. Solve $\frac{1}{3}x + \frac{1}{4}x = \frac{5}{6}$.
If you missed this problem, review **Example 3.49**.
3. Solve: $18t^2 - 30 = -33t$.
If you missed this problem, review **Example 7.79**.

Solve Uniform Motion Applications

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate •	Time =	Distance

The formula $D = rt$ assumes we know *r* and *t* and use them to find D. If we know D and *r* and need to find *t*, we would solve the equation for *t* and get the formula $t = \frac{D}{r}$.

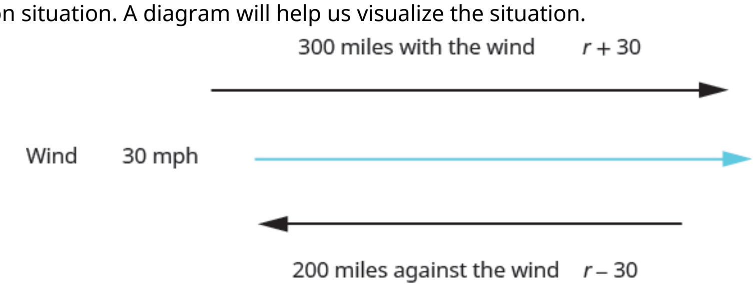
We have also explained how flying with or against a current affects the speed of a vehicle. We will revisit that idea in the next example.

EXAMPLE 8.81

An airplane can fly 200 miles into a 30 mph headwind in the same amount of time it takes to fly 300 miles with a 30 mph tailwind. What is the speed of the airplane?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the airplane.	Let $r =$ the speed of the airplane.
When the plane flies with the wind, the wind increases its speed and the rate is $r + 30$.	
When the plane flies against the wind, the wind decreases its speed and the rate is $r - 30$.	
Write in the rates. Write in the distances. Since $D = r \cdot t$, we solve for *t* and get $\frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	(see table below)
We know the times are equal and so we write our equation.	$\frac{200}{r-30} = \frac{300}{r+30}$
We multiply both sides by the LCD. $200(r+30) = 300(r-30)$	$(r+30)(r-30)\left(\frac{200}{r-30}\right) = (r+30)(r-30)\left(\frac{300}{r+30}\right)$
Simplify.	$(r+30)(200) = (r-30)(300)$
	$200r + 6000 = 300r - 9000$
Solve.	$15000 = 100r$ $150 = r$
Check.	
Is 150 mph a reasonable speed for an airplane? Yes. If the plane is traveling 150 mph and the wind is 30 mph:	
Tailwind $150 + 30 = 180\text{mph}$ $\frac{300}{180} = \frac{5}{3}$ hours	
Headwind $150 - 30 = 120\text{mph}$ $\frac{200}{120} = \frac{5}{3}$ hours	
The times are equal, so it checks.	The plane was traveling 150 mph.

	Rate •	Time =	Distance
Headwind	$r - 30$	$\frac{200}{r-30}$	200
Tailwind	$r + 30$	$\frac{300}{r+30}$	300

TRY IT : : 8.161

Link can ride his bike 20 miles into a 3 mph headwind in the same amount of time he can ride 30 miles with a 3 mph tailwind. What is Link's biking speed?

TRY IT : : 8.162

Judy can sail her boat 5 miles into a 7 mph headwind in the same amount of time she can sail 12 miles with a 7 mph tailwind. What is the speed of Judy's boat without a wind?

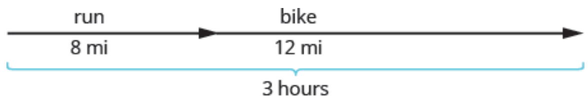
In the next example, we will know the total time resulting from travelling different distances at different speeds.

EXAMPLE 8.82

Jazmine trained for 3 hours on Saturday. She ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. What is her running speed?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Jazmine's running speed.	Let $r =$ Jazmine's running speed.
Her biking speed is 4 miles faster than her running speed.	$r + 4 =$ her biking speed
The distances are given, enter them into the chart.	
Since $D = r \cdot t$, we solve for *t* and get $t = \frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	see chart below
Write a word sentence.	Her time plus the time biking is 3 hours.
Translate the sentence to get the equation.	$\frac{8}{r} + \frac{24}{r+4} = 3$
Solve.	$\begin{aligned} r(r+4)\left(\frac{8}{r} + \frac{24}{r+4}\right) &= 3 \cdot r(r+4) \\ 8(r+4) + 24r &= 3r(r+4) \\ 8r + 32 + 24r &= 3r^2 + 12r \\ 32 + 32r &= 3r^2 + 12r \\ 0 &= 3r^2 - 20r - 32 \\ 0 &= (3r+4)(r-8) \end{aligned}$
	$(3r+4) = 0 \quad (r-8) = 0$
	$r = -\frac{4}{3} \quad r = 8$
Check. $\cancel{r = -\frac{4}{3}}$ $\quad r = 8$	
A negative speed does not make sense in this problem, so $r = 8$ is the solution.	
Is 8 mph a reasonable running speed? Yes.	

	Rate	• Time	= Distance
Run	r	$\frac{8}{r}$	8
Bike	$r+4$	$\frac{24}{r+4}$	24
		3	

Run 8 mph $\quad \frac{8 \text{ miles}}{8 \text{ mph}} = 1$ hour

Bike 12 mph $\quad \frac{24 \text{ miles}}{12 \text{ mph}} = 2$ hours

Total 3 hours

Jazmine's running speed is 8 mph.

TRY IT : : 8.163

Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhill speed. What was Dennis' speed going uphill and his speed going downhill?

TRY IT : : 8.164

Tony drove 4 hours to his home, driving 208 miles on the interstate and 40 miles on country roads. If he drove 15 mph faster on the interstate than on the country roads, what was his rate on the country roads?

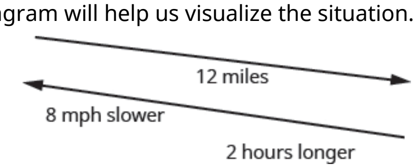
Once again, we will use the uniform motion formula solved for the variable *t*.

EXAMPLE 8.83

Hamilton rode his bike downhill 12 miles on the river trail from his house to the ocean and then rode uphill to return home. His uphill speed was 8 miles per hour slower than his downhill speed. It took him 2 hours longer to get home than it took him to get to the ocean. Find Hamilton's downhill speed.

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Hamilton's downhill speed.

Let $r =$ Hamilton's downhill speed.

His uphill speed is 8 miles per hour slower. Enter the rates into the chart.

$h - 8 =$ Hamilton's uphill speed

The distance is the same in both directions, 12 miles.

Since $D = r \bullet t$, we solve for *t* and get $t = \frac{D}{r}$.

We divide the distance by the rate in each row, and place the expression in the time column.

	Rate	• Time	= Distance
Downhill	h	$\frac{12}{h}$	12
Uphill	$h-8$	$\frac{12}{h-8}$	12

Write a word sentence about the time.

He took 2 hours longer uphill than downhill. The uphill time is 2 more than the downhill time.

Translate the sentence to get the equation.

$$\frac{12}{h-8} = \frac{12}{h} + 2$$

Solve.

$$\begin{aligned}
h(h-8)\left(\frac{12}{h-8}\right) &= h(h-8)\left(\frac{12}{h}+2\right) \\
12h &= 12(h-8) + 2h(h-8) \\
12h &= 12h - 96 + 2h^2 - 16h \\
0 &= 2h^2 - 16h - 96 \\
0 &= 2\left(h^2 - 8h - 48\right) \\
0 &= 2(h-12)(h+4) \\
h - 12 &= 0 \quad h + 4 = 0 \\
h &= 12 \qquad \cancel{h = 4}
\end{aligned}$$

Check. Is 12 mph a reasonable speed for biking downhill? Yes.

Downhill	12 mph	$\frac{12 \text{ miles}}{12 \text{ mph}} = 1 \text{ hour}$	
Uphill	$12 - 8 = 4$ mph	$\frac{12 \text{ miles}}{4 \text{ mph}} = 3 \text{ hours}$	
			The uphill time is 2 hours more than the downhill time. Hamilton's downhill speed is 12 mph.

TRY IT : : 8.165

Kayla rode her bike 75 miles home from college one weekend and then rode the bus back to college. It took her 2 hours less to ride back to college on the bus than it took her to ride home on her bike, and the average speed of the bus was 10 miles per hour faster than Kayla's biking speed. Find Kayla's biking speed.

TRY IT : : 8.166

Victoria jogs 12 miles to the park along a flat trail and then returns by jogging on an 18 mile hilly trail. She jogs 1 mile per hour slower on the hilly trail than on the flat trail, and her return trip takes her two hours longer. Find her rate of jogging on the flat trail.

Solve Work Applications

Suppose Pete can paint a room in 10 hours. If he works at a steady pace, in 1 hour he would paint $\frac{1}{10}$ of the room. If Alicia would take 8 hours to paint the same room, then in 1 hour she would paint $\frac{1}{8}$ of the room. How long would it take Pete and Alicia to paint the room if they worked together (and didn't interfere with each other's progress)?

This is a typical 'work' application. There are three quantities involved here – the time it would take each of the two people to do the job alone and the time it would take for them to do the job together.

Let's get back to Pete and Alicia painting the room. We will let t be the number of hours it would take them to paint the room together. So in 1 hour working together they have completed $\frac{1}{t}$ of the job.

	Rate	• Time	= Distance
Downhill	h	$\frac{12}{h}$	12
Uphill	$h-8$	$\frac{12}{h-8}$	12

In one hour Pete did $\frac{1}{10}$ of the job. Alicia did $\frac{1}{8}$ of the job. And together they did $\frac{1}{t}$ of the job.

We can model this with the word equation and then translate to a rational equation. To find the time it would take them if they worked together, we solve for t.

$$\underbrace{\text{Pete's part}}_{\frac{1}{10}} + \underbrace{\text{Alicia's part}}_{\frac{1}{8}} = \underbrace{\text{part of total}}_{\frac{1}{t}}$$

$$\frac{1}{10} + \frac{1}{8} = \frac{1}{t}$$

$$\frac{1}{10} + \frac{1}{8} = \frac{1}{t}$$

Multiply by the LCD, $40t$.	$40t\left(\frac{1}{10}+\frac{1}{8}\right)=40t\left(\frac{1}{t}\right)$
Distribute.	$40t\cdot\frac{1}{10}+40t\cdot\frac{1}{8}=40t\left(\frac{1}{t}\right)$
Simplify and solve.	$4t+5t=40$
	$9t=40$
	$t=\frac{40}{9}$
We'll write as a mixed number so that we can convert it to hours and minutes.	$t=4\frac{4}{9}$ hours
Remember, 1 hour = 60 minutes.	$t=4$ hours $+\frac{4}{9}$(60 minutes)
Multiply, and then round to the nearest minute.	$t=4$ hours $+ 27$ minutes
	It would take Pete and Alica about 4 hours and 27 minutes to paint the room.

Keep in mind, it should take less time for two people to complete a job working together than for either person to do it alone.

EXAMPLE 8.84

The weekly gossip magazine has a big story about the Princess' baby and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours to do the job and Press #2 takes 12 hours to do the job. How long will it take the printer to get the magazine printed with both presses running together?

Solution

This is a work problem. A chart will help us organize the information.

Let $t =$ the number of hours needed to complete the job together.	
Enter the hours per job for Press #1, Press #2 and when they work together. If a job on Press #1 takes 6 hours, then in 1 hour $\frac{1}{6}$ of the job is completed. Similarly find the part of the job completed/hours for Press #2 and when they both work together.	see table below
Write a word sentence.	
The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.	
Translate to an equation.	Work completed by $\underbrace{\text{Press \#1} + \text{Press \#2} = \text{Together}}$ $\frac{1}{6} \quad + \quad \frac{1}{12} \quad = \quad \frac{1}{t}$
Solve.	$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$
Multiply by the LCD, $12t$.	$12t\left(\frac{1}{6} + \frac{1}{12}\right) = 12t\left(\frac{1}{t}\right)$
Simplify.	$2t + t = 12$
	$3t = 12$
	$t = 4$
	When both presses are running it takes 4 hours to do the job.

	Number of hours to complete the job	Part of job completed/hour
Press #1	6	$\frac{1}{6}$
Press #2	12	$\frac{1}{12}$
Together	t	$\frac{1}{t}$

TRY IT : : 8.167

One gardener can mow a golf course in 4 hours, while another gardener can mow the same golf course in 6 hours. How long would it take if the two gardeners worked together to mow the golf course?

TRY IT : : 8.168

Carrie can weed the garden in 7 hours, while her mother can do it in 3. How long will it take the two of them working together?

EXAMPLE 8.85

Corey can shovel all the snow from the sidewalk and driveway in 4 hours. If he and his twin Casey work together, they can finish shoveling the snow in 2 hours. How many hours would it take Casey to do the job by himself?

Solution

This is a work application. A chart will help us organize the information.

We are looking for how many hours it would take Casey to complete the job by himself.

Let $t =$ the number of hours needed for Casey to complete.

Enter the hours per job for Corey, Casey, and when they work together.
If Corey takes 4 hours, then in 1 hour $\frac{1}{4}$ of the job is completed. Similarly find the part of the job completed/hours for Casey and when they both work together.

	Number of hours needed to complete the job	Part of job completed/hour
Corey	4	$\frac{1}{4}$
Casey	t	$\frac{1}{t}$
Together	2	$\frac{1}{2}$

Write a word sentence.

The part completed by Corey plus the part completed by Casey equals the amount completed together.

Translate to an equation:

$$\underbrace{\text{Corey} + \text{Casey} = \text{Together}}$$
$$\frac{1}{4} + \frac{1}{t} = \frac{1}{2}$$

Solve.

$$\frac{1}{4} + \frac{1}{t} = \frac{1}{2}$$

Multiply by the LCD, $4t$.

$$4t\left(\frac{1}{4} + \frac{1}{t}\right) = 4t\left(\frac{1}{2}\right)$$

Simplify.

$$t + 4 = 2t$$

$$4 = t$$

It would take Casey 4 hours to do the job alone.

TRY IT : : 8.169

Two hoses can fill a swimming pool in 10 hours. It would take one hose 26 hours to fill the pool by itself. How long would it take for the other hose, working alone, to fill the pool?

TRY IT : : 8.170

Cara and Cindy, working together, can rake the yard in 4 hours. Working alone, it takes Cindy 6 hours to rake the yard. How long would it take Cara to rake the yard alone?

8.8 EXERCISES

Practice Makes Perfect

Solve Uniform Motion Applications

In the following exercises, solve uniform motion applications

429. Mary takes a sightseeing tour on a helicopter that can fly 450 miles against a 35 mph headwind in the same amount of time it can travel 702 miles with a 35 mph tailwind. Find the speed of the helicopter.

430. A private jet can fly 1210 miles against a 25 mph headwind in the same amount of time it can fly 1694 miles with a 25 mph tailwind. Find the speed of the jet.

431. A boat travels 140 miles downstream in the same time as it travels 92 miles upstream. The speed of the current is 6mph. What is the speed of the boat?

432. Darrin can skateboard 2 miles against a 4 mph wind in the same amount of time he skateboards 6 miles with a 4 mph wind. Find the speed Darrin skateboards with no wind.

433. Jane spent 2 hours exploring a mountain with a dirt bike. When she rode the 40 miles uphill, she went 5 mph slower than when she reached the peak and rode for 12 miles along the summit. What was her rate along the summit?

434. Jill wanted to lose some weight so she planned a day of exercising. She spent a total of 2 hours riding her bike and jogging. She biked for 12 miles and jogged for 6 miles. Her rate for jogging was 10 mph less than biking rate. What was her rate when jogging?

435. Bill wanted to try out different water craft. He went 62 miles downstream in a motor boat and 27 miles downstream on a jet ski. His speed on the jet ski was 10 mph faster than in the motor boat. Bill spent a total of 4 hours on the water. What was his rate of speed in the motor boat?

436. Nancy took a 3 hour drive. She went 50 miles before she got caught in a storm. Then she drove 68 miles at 9 mph less than she had driven when the weather was good. What was her speed driving in the storm?

437. Chester rode his bike uphill 24 miles and then back downhill at 2 mph faster than his uphill. If it took him 2 hours longer to ride uphill than downhill, l, what was his uphill rate?

438. Matthew jogged to his friend's house 12 miles away and then got a ride back home. It took him 2 hours longer to jog there than ride back. His jogging rate was 25 mph slower than the rate when he was riding. What was his jogging rate?

439. Hudson travels 1080 miles in a jet and then 240 miles by car to get to a business meeting. The jet goes 300 mph faster than the rate of the car, and the car ride takes 1 hour longer than the jet. What is the speed of the car?

440. Nathan walked on an asphalt pathway for 12 miles. He walked the 12 miles back to his car on a gravel road through the forest. On the asphalt he walked 2 miles per hour faster than on the gravel. The walk on the gravel took one hour longer than the walk on the asphalt. How fast did he walk on the gravel?

441. John can fly his airplane 2800 miles with a wind speed of 50 mph in the same time he can travel 2400 miles against the wind. If the speed of the wind is 50 mph, find the speed of his airplane.

442. Jim's speedboat can travel 20 miles upstream against a 3 mph current in the same amount of time it travels 22 miles downstream with a 3 mph current speed. Find the speed of the Jim's boat.

443. Hazel needs to get to her granddaughter's house by taking an airplane and a rental car. She travels 900 miles by plane and 250 miles by car. The plane travels 250 mph faster than the car. If she drives the rental car for 2 hours more than she rode the plane, find the speed of the car.

444. Stu trained for 3 hours yesterday. He ran 14 miles and then biked 40 miles. His biking speed is 6 mph faster than his running speed. What is his running speed?

445. When driving the 9 hour trip home, Sharon drove 390 miles on the interstate and 150 miles on country roads. Her speed on the interstate was 15 more than on country roads. What was her speed on country roads?

446. Two sisters like to compete on their bike rides. Tamara can go 4 mph faster than her sister, Samantha. If it takes Samantha 1 hours longer than Tamara to go 80 miles, how fast can Samantha ride her bike?

Solve Work Applications

In the following exercises, solve work applications.

447. Mike, an experienced bricklayer, can build a wall in 3 hours, while his son, who is learning, can do the job in 6 hours. How long does it take for them to build a wall together?

448. It takes Sam 4 hours to rake the front lawn while his brother, Dave, can rake the lawn in 2 hours. How long will it take them to rake the lawn working together?

449. Mary can clean her apartment in 6 hours while her roommate can clean the apartment in 5 hours. If they work together, how long would it take them to clean the apartment?

450. Brian can lay a slab of concrete in 6 hours, while Greg can do it in 4 hours. If Brian and Greg work together, how long will it take?

451. Leeson can proofread a newspaper copy in 4 hours. If Ryan helps, they can do the job in 3 hours. How long would it take for Ryan to do his job alone?

452. Paul can clean a classroom floor in 3 hours. When his assistant helps him, the job takes 2 hours. How long would it take the assistant to do it alone?

453. Josephine can correct her students' test papers in 5 hours, but if her teacher's assistant helps, it would take them 3 hours. How long would it take the assistant to do it alone?

454. Washing his dad's car alone, eight year old Levi takes 2.5 hours. If his dad helps him, then it takes 1 hour. How long does it take the Levi's dad to wash the car by himself?

455. Jackson can remove the shingles off of a house in 7 hours, while Martin can remove the shingles in 5 hours. How long will it take them to remove the shingles if they work together?

456. At the end of the day Dodie can clean her hair salon in 15 minutes. Ann, who works with her, can clean the salon in 30 minutes. How long would it take them to clean the shop if they work together?

457. Ronald can shovel the driveway in 4 hours, but if his brother Donald helps it would take 2 hours. How long would it take Donald to shovel the driveway alone?

458. It takes Tina 3 hours to frost her holiday cookies, but if Candy helps her it takes 2 hours. How long would it take Candy to frost the holiday cookies by herself?

Everyday Math

459. Dana enjoys taking her dog for a walk, but sometimes her dog gets away and she has to run after him. Dana walked her dog for 7 miles but then had to run for 1 mile, spending a total time of 2.5 hours with her dog. Her running speed was 3 mph faster than her walking speed. Find her walking speed.

460. Ken and Joe leave their apartment to go to a football game 45 miles away. Ken drives his car 30 mph faster Joe can ride his bike. If it takes Joe 2 hours longer than Ken to get to the game, what is Joe's speed?

Writing Exercises

461. In **Example 8.83**, the solution $h=-4$ is crossed out. Explain why.

462. Paula and Yuki are roommates. It takes Paula 3 hours to clean their apartment. It takes Yuki 4 hours to clean the apartment. The equation $\frac{1}{3}+\frac{1}{4}=\frac{1}{t}$ can be used to find *t*, the number of hours it would take both of them, working together, to clean their apartment. Explain how this equation models the situation.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
solve uniform motion applications.			
solve work applications.			

ⓑ *On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

8.9 Use Direct and Inverse Variation

Learning Objectives

By the end of this section, you will be able to:

- Solve direct variation problems
- Solve inverse variation problems

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Find the multiplicative inverse of -8.
   If you missed this problem, review **Example 1.126**.
2. Solve for n: $45 = 20n$.
   If you missed this problem, review **Example 2.13**.
3. Evaluate $5x^2$ when $x = 10$.
   If you missed this problem, review **Example 1.20**.

When two quantities are related by a proportion, we say they are *proportional* to each other. Another way to express this relation is to talk about the *variation* of the two quantities. We will discuss direct variation and inverse variation in this section.

Solve Direct Variation Problems

Lindsay gets paid $15 per hour at her job. If we let *s* be her salary and *h* be the number of hours she has worked, we could model this situation with the equation

$$s = 15h$$

Lindsay's salary is the product of a constant, 15, and the number of hours she works. We say that Lindsay's salary *varies directly* with the number of hours she works. Two variables vary directly if one is the product of a constant and the other.

Direct Variation

For any two variables *x* and *y*, *y* varies directly with *x* if

$$y = kx, \quad \text{where } k \neq 0$$

The constant *k* is called the constant of variation.

In applications using direct variation, generally we will know values of one pair of the variables and will be asked to find the equation that relates *x* and *y*. Then we can use that equation to find values of *y* for other values of *x*.

EXAMPLE 8.86 HOW TO SOLVE DIRECT VARIATION PROBLEMS

If y varies directly with x and $y = 20$ when $x = 8$, find the equation that relates *x* and *y*.

Solution

Step 1. Write the formula for direct variation.	The direct variation formula is $y = kx$.	$y = kx$
Step 2. Substitute the given values for the variables.	We are given $y = 20$, $x = 8$.	$20 = k \cdot 8$
Step 3. Solve for the constant of variation.	Divide both sides of the equation by 8, then simplify.	$\frac{20}{8} = k$ $k = 2.5$
Step 4. Write the equation that relates *x* and *y*.	Rewrite the general equation with the value we found for *k*.	$y = 2.5x$

TRY IT : : 8.171 If y varies directly as x and $y = 3,$ when $x = 10.$ find the equation that relates x and y.

TRY IT : : 8.172 If y varies directly as x and $y = 12$ when $x = 4$ find the equation that relates x and y.

We'll list the steps below.

HOW TO : : SOLVE DIRECT VARIATION PROBLEMS.

Step 1. Write the formula for direct variation.

Step 2. Substitute the given values for the variables.

Step 3. Solve for the constant of variation.

Step 4. Write the equation that relates x and y.

Now we'll solve a few applications of direct variation.

EXAMPLE 8.87

When Raoul runs on the treadmill at the gym, the number of calories, c, he burns varies directly with the number of minutes, m, he uses the treadmill. He burned 315 calories when he used the treadmill for 18 minutes.

ⓐ Write the equation that relates c and m.

ⓑ How many calories would he burn if he ran on the treadmill for 25 minutes?

Solution

ⓐ

	The number of calories, c, varies directly with the number of minutes, m, on the treadmill, and $c = 315$ when $m = 18$.
Write the formula for direct variation.	$y = kx$
We will use c in place of y and m in place of x.	$c = km$
Substitute the given values for the variables.	$315 = k \cdot 18$
Solve for the constant of variation.	$\frac{315}{18} = \frac{k \cdot 18}{18}$
	$17.5 = k$
Write the equation that relates c and m.	$c = km$
Substitute in the constant of variation.	$c = 17.5m$

ⓑ

	Find c when $m = 25$.
Write the equation that relates c and m.	$c = 17.5m$
Substitute the given value for m.	$c = 17.5(25)$
Simplify.	$c = 437.5$

Raoul would burn 437.5 calories if he used the treadmill for 25 minutes.

TRY IT : : 8.173

The number of calories, *c*, burned varies directly with the amount of time, *t,* spent exercising. Arnold burned 312 calories in 65 minutes exercising.

ⓐ Write the equation that relates *c* and *t*.

ⓑ How many calories would he burn if he exercises for 90 minutes?

TRY IT : : 8.174

The distance a moving body travels, *d*, varies directly with time, *t*, it moves. A train travels 100 miles in 2 hours

ⓐ Write the equation that relates *d* and *t*.

ⓑ How many miles would it travel in 5 hours?

In the previous example, the variables *c* and *m* were named in the problem. Usually that is not the case. We will have to name the variables in the next example as part of the solution, just like we do in most applied problems.

EXAMPLE 8.88

The number of gallons of gas Eunice's car uses varies directly with the number of miles she drives. Last week she drove 469.8 miles and used 14.5 gallons of gas.

ⓐ Write the equation that relates the number of gallons of gas used to the number of miles driven.

ⓑ How many gallons of gas would Eunice's car use if she drove 1000 miles?

Solution

	The number of gallons of gas varies directly with the number of miles driven.
First we will name the variables.	Let $g =$ number of gallons of gas. $m =$ number of miles driven
Write the formula for direct variation.	$y = kx$
We will use g in place of y and m in place of x.	$g = km$
Substitute the given values for the variables.	$g = 14.5$ when $m = 469.8$
	$14.5 = k(469.8)$
Solve for the constant of variation.	$\frac{14.5}{469.8} = \frac{k(469.8)}{469.8}$
We will round to the nearest thousandth.	$0.031 = k$
Write the equation that relates g and m.	$g = km$
Substitute in the constant of variation.	$g = 0.031m$

ⓑ

	Find g when $m = 1000$.
Write the equation that relates g and m.	$g = 0.031m$
Substitute the given value for m.	$g = 0.031(1000)$
Simplify.	$g = 31$
	Eunice's car would use 31 gallons of gas if she drove it 1,000 miles.

Notice that in this example, the units on the constant of variation are gallons/mile. In everyday life, we usually talk about miles/gallon.

TRY IT : : 8.175

The distance that Brad travels varies directly with the time spent traveling. Brad travelled 660 miles in 12 hours,

ⓐ Write the equation that relates the number of miles travelled to the time.

ⓑ How many miles could Brad travel in 4 hours?

TRY IT : : 8.176

The weight of a liquid varies directly as its volume. A liquid that weighs 24 pounds has a volume of 4 gallons.

ⓐ Write the equation that relates the weight to the volume.

ⓑ If a liquid has volume 13 gallons, what is its weight?

In some situations, one variable varies directly with the square of the other variable. When that happens, the equation of direct variation is $y = kx^2$. We solve these applications just as we did the previous ones, by substituting the given values into the equation to solve for *k*.

EXAMPLE 8.89

The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 4" will support a maximum load of 75 pounds.

ⓐ Write the equation that relates the maximum load to the cross-section.

ⓑ What is the maximum load that can be supported by a beam with diagonal 8"?

Solution

	The maximum load varies directly with the square of the diagonal of the cross-section.
Name the variables.	Let $L =$ maximum load. $c =$ the diagonal of the cross-section
Write the formula for direct variation, where y varies directly with the square of x.	$y = kx^2$
We will use L in place of y and c in place of x.	$L = kc^2$
Substitute the given values for the variables.	$L = 75$ when $c = 4$
	$75 = k \cdot 4^2$

Solve for the constant of variation.	$\frac{75}{16} = \frac{k \cdot 16}{16}$
	$4.6875 = k$
Write the equation that relates L and c.	$L = kc^2$
Substitute in the constant of variation.	$L = 4.6875c^2$

ⓑ

	Find L when $c = 8$.
Write the equation that relates L and c.	$L = 4.6875c^2$
Substitute the given value for c.	$L = 4.6875(8)^2$
Simplify.	$L = 300$
	A beam with diagonal 8" could support a maximum load of 300 pounds.

TRY IT : : 8.177

The distance an object falls is directly proportional to the square of the time it falls. A ball falls 144 feet in 3 seconds.

ⓐ Write the equation that relates the distance to the time.
ⓑ How far will an object fall in 4 seconds?

TRY IT : : 8.178

The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.

ⓐ Write the equation that relates the area to the radius.

ⓑ What is the area of a pizza with a radius of 9 inches?

Solve Inverse Variation Problems

Many applications involve two variable that *vary inversely*. As one variable increases, the other decreases. The equation that relates them is $y = \frac{k}{x}$.

Inverse Variation

For any two variables *x* and *y*, *y* varies inversely with *x* if

$$y = \frac{k}{x}, \text{ where } k \neq 0$$

The constant *k* is called the constant of variation.

The word 'inverse' in inverse variation refers to the multiplicative inverse. The multiplicative inverse of *x* is $\frac{1}{x}$.

We solve inverse variation problems in the same way we solved direct variation problems. Only the general form of the equation has changed. We will copy the procedure box here and just change 'direct' to 'inverse'.

HOW TO : : SOLVE INVERSE VARIATION PROBLEMS.

Step 1. Write the formula for inverse variation.
Step 2. Substitute the given values for the variables.
Step 3. Solve for the constant of variation.
Step 4. Write the equation that relates x and y.

EXAMPLE 8.90

If *y* varies inversely with x and $y = 20$ when $x = 8$, find the equation that relates *x* and *y*.

Solution

Write the formula for inverse variation.	$y = \frac{k}{x}$
Substitute the given values for the variables.	$y = 20$ when $x = 8$
	$20 = \frac{k}{8}$
Solve for the constant of variation.	$8(20) = 8\left(\frac{k}{8}\right)$
	$160 = k$
Write the equation that relates x and y.	$y = \frac{k}{x}$
Substitute in the constant of variation.	$y = \frac{160}{x}$

TRY IT : : 8.179

If p varies inversely with q and $p = 30$ when $q = 12$ find the equation that relates p and q.

TRY IT : : 8.180

If y varies inversely with x and $y = 8$ when $x = 2$ find the equation that relates x and y.

EXAMPLE 8.91

The fuel consumption (mpg) of a car varies inversely with its weight. A car that weighs 3100 pounds gets 26 mpg on the highway.

ⓐ Write the equation of variation.

ⓑ What would be the fuel consumption of a car that weighs 4030 pounds?

Solution

ⓐ

	The fuel consumption varies inversely with the weight.

First we will name the variables.	Let $f =$ fuel consumption. $w =$ weight
Write the formula for inverse variation.	$y = \frac{k}{x}$
We will use f in place of y and w in place of x.	$f = \frac{k}{w}$
Substitute the given values for the variables.	$f = 26$ when $w = 3100$
	$26 = \frac{k}{3100}$
Solve for the constant of variation.	$3100(26) = 3100\left(\frac{k}{3100}\right)$
	$80{,}600 = k$
Write the equation that relates f and w.	$f = \frac{k}{w}$
Substitute in the constant of variation.	$f = \frac{80{,}600}{w}$

ⓑ

	Find f when $w = 4030$.
Write the equation that relates f and w.	$f = \frac{80{,}600}{w}$
Substitute the given value for w.	$f = \frac{80{,}600}{4030}$
Simplify.	$f = 20$
	A car that weighs 4030 pounds would have fuel consumption of 20 mpg.

TRY IT : : 8.181

A car's value varies inversely with its age. Elena bought a two-year-old car for $20,000.

ⓐ Write the equation of variation.
ⓑ What will be the value of Elena's car when it is 5 years old?

TRY IT : : 8.182

The time required to empty a pool varies inversely as the rate of pumping. It took Lucy 2.5 hours to empty her pool using a pump that was rated at 400 gpm (gallons per minute).

ⓐ Write the equation of variation.

ⓑ How long will it take her to empty the pool using a pump rated at 500 gpm?

EXAMPLE 8.92

The frequency of a guitar string varies inversely with its length. A 26” long string has a frequency of 440 vibrations per second.

ⓐ Write the equation of variation.

ⓑ How many vibrations per second will there be if the string's length is reduced to 20” by putting a finger on a fret?

Solution

ⓐ

	The frequency varies inversely with the length.
Name the variables.	Let $f =$ frequency. $L =$ length
Write the formula for inverse variation.	$y = \frac{k}{x}$
We will use f in place of y and L in place of x.	$f = \frac{k}{L}$
Substitute the given values for the variables.	$f = 440$ when $L = 26$
	$440 = \frac{k}{26}$
Solve for the constant of variation.	$26(440) = 26\left(\frac{k}{26}\right)$
	$11{,}440 = k$
Write the equation that relates f and L.	$f = \frac{k}{L}$
Substitute in the constant of variation.	$f = \frac{11{,}440}{L}$

ⓑ

	Find f when $L = 20$.
Write the equation that relates f and L.	$f = \frac{11{,}440}{L}$
Substitute the given value for L.	$f = \frac{11{,}440}{20}$
Simplify.	$f = 572$
	A 20” guitar string has frequency 572 vibrations per second.

TRY IT : : 8.183

The number of hours it takes for ice to melt varies inversely with the air temperature. Suppose a block of ice melts in 2 hours when the temperature is 65 degrees.

ⓐ Write the equation of variation.

ⓑ How many hours would it take for the same block of ice to melt if the temperature was 78 degrees?

TRY IT : : 8.184

The force needed to break a board varies inversely with its length. Richard uses 24 pounds of pressure to break a 2-foot long board.

ⓐ Write the equation of variation.

ⓑ How many pounds of pressure is needed to break a 5-foot long board?

8.9 EXERCISES

Practice Makes Perfect

Solve Direct Variation Problems

In the following exercises, solve.

463. If y varies directly as x and $y = 14,$ when $x = 3$, find the equation that relates x and y.

464. If p varies directly as q and $p = 5,$ when $q = 2$, find the equation that relates p and q.

465. If v varies directly as w and $v = 24,$ when $w = 8$, find the equation that relates v and w.

466. If a varies directly as b and $a = 16,$ when $b = 4$, find the equation that relates a and b.

467. If p varies directly as q and $p = 9.6,$ when $q = 3$, find the equation that relates p and q.

468. If y varies directly as x and $y = 12.4,$ when $x = 4,$ find the equation that relates x and y

469. If a varies directly as b and $a = 6,$ when $b = \frac{1}{3}$, find the equation that relates a and b.

470. If v varies directly as w and $v = 8,$ when $w = \frac{1}{2}$, find the equation that relates v and w.

471. The amount of money Sally earns, *P*, varies directly with the number, *n*, of necklaces she sells. When Sally sells 15 necklaces she earns $150.

ⓐ Write the equation that relates *P* and *n*.

ⓑ How much money would she earn if she sold 4 necklaces?

472. The price, *P*, that Eric pays for gas varies directly with the number of gallons, *g*, he buys. It costs him $50 to buy 20 gallons of gas.

ⓐ Write the equation that relates *P* and *g*.

ⓑ How much would 33 gallons cost Eric?

473. Terri needs to make some pies for a fundraiser. The number of apples, *a*, varies directly with number of pies, *p*. It takes nine apples to make two pies.

ⓐ Write the equation that relates *a* and *p*.

ⓑ How many apples would Terri need for six pies?

474. Joseph is traveling on a road trip. The distance, *d*, he travels before stopping for lunch varies directly with the speed, *v*, he travels. He can travel 120 miles at a speed of 60 mph.

ⓐ Write the equation that relates *d* and *v*.

ⓑ How far would he travel before stopping for lunch at a rate of 65 mph?

475. The price of gas that Jesse purchased varies directly to how many gallons he purchased. He purchased 10 gallons of gas for $39.80.

ⓐ Write the equation that relates the price to the number of gallons.

ⓑ How much will it cost Jesse for 15 gallons of gas?

476. The distance that Sarah travels varies directly to how long she drives. She travels 440 miles in 8 hours.

ⓐ Write the equation that relates the distance to the number of hours.

ⓑ How far can Sally travel in 6 hours?

477. The mass of a liquid varies directly with its volume. A liquid with mass 16 kilograms has a volume of 2 liters.

ⓐ Write the equation that relates the mass to the volume.

ⓑ What is the volume of this liquid if its mass is 128 kilograms?

478. The length that a spring stretches varies directly with a weight placed at the end of the spring. When Sarah placed a 10 pound watermelon on a hanging scale, the spring stretched 5 inches.

ⓐ Write the equation that relates the length of the spring to the weight.

ⓑ What weight of watermelon would stretch the spring 6 inches?

479. The distance an object falls varies directly to the square of the time it falls. A ball falls 45 feet in 3 seconds.

ⓐ Write the equation that relates the distance to the time.

ⓑ How far will the ball fall in 7 seconds?

480. The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 6 inch will support a maximum load of 108 pounds.

ⓐ Write the equation that relates the load to the diagonal of the cross-section.

ⓑ What load will a beam with a 10 inch diagonal support?

481. The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.

ⓐ Write the equation that relates the area to the radius.

ⓑ What is the area of a personal pizza with a radius 4 inches?

482. The distance an object falls varies directly to the square of the time it falls. A ball falls 72 feet in 3 seconds,

ⓐ Write the equation that relates the distance to the time.

ⓑ How far will the ball have fallen in 8 seconds?

Solve Inverse Variation Problems

In the following exercises, solve.

483. If y varies inversely with x and $y = 5$ when $x = 4$ find the equation that relates x and y.

484. If p varies inversely with q and $p = 2$ when $q = 1$ find the equation that relates p and q.

485. If v varies inversely with w and $v = 6$ when $w = \frac{1}{2}$ find the equation that relates v and w.

486. If a varies inversely with b and $a = 12$ when $b = \frac{1}{3}$ find the equation that relates a and b.

Write an inverse variation equation to solve the following problems.

487. The fuel consumption (mpg) of a car varies inversely with its weight. A Toyota Corolla weighs 2800 pounds and gets 33 mpg on the highway.

ⓐ Write the equation that relates the mpg to the car's weight.

ⓑ What would the fuel consumption be for a Toyota Sequoia that weighs 5500 pounds?

488. A car's value varies inversely with its age. Jackie bought a 10 year old car for $2,400.

ⓐ Write the equation that relates the car's value to its age.

ⓑ What will be the value of Jackie's car when it is 15 years old ?

489. The time required to empty a tank varies inversely as the rate of pumping. It took Janet 5 hours to pump her flooded basement using a pump that was rated at 200 gpm (gallons per minute),

ⓐ Write the equation that relates the number of hours to the pump rate.

ⓑ How long would it take Janet to pump her basement if she used a pump rated at 400 gpm?

490. The volume of a gas in a container varies inversely as pressure on the gas. A container of helium has a volume of 370 cubic inches under a pressure of 15 psi.

ⓐ Write the equation that relates the volume to the pressure.

ⓑ What would be the volume of this gas if the pressure was increased to 20 psi?

491. On a string instrument, the length of a string varies inversely as the frequency of its vibrations. An 11-inch string on a violin has a frequency of 400 cycles per second.

ⓐ Write the equation that relates the string length to its frequency.

ⓑ What is the frequency of a 10-inch string?

492. Paul, a dentist, determined that the number of cavities that develops in his patient's mouth each year varies inversely to the number of minutes spent brushing each night. His patient, Lori, had 4 cavities when brushing her teeth 30 seconds (0.5 minutes) each night.

ⓐ Write the equation that relates the number of cavities to the time spent brushing.

ⓑ How many cavities would Paul expect Lori to have if she had brushed her teeth for 2 minutes each night?

493. The number of tickets for a sports fundraiser varies inversely to the price of each ticket. Brianna can buy 25 tickets at $5each.

ⓐ Write the equation that relates the number of tickets to the price of each ticket.

ⓑ How many tickets could Brianna buy if the price of each ticket was $2.50?

494. Boyle's Law states that if the temperature of a gas stays constant, then the pressure varies inversely to the volume of the gas. Braydon, a scuba diver, has a tank that holds 6 liters of air under a pressure of 220 psi.

ⓐ Write the equation that relates pressure to volume.

ⓑ If the pressure increases to 330 psi, how much air can Braydon's tank hold?

Mixed Practice

495. If y varies directly as x and $y = 5,$ when $x = 3.$, find the equation that relates x and y.

496. If v varies directly as w and $v = 21,$ when $w = 8.$ find the equation that relates v and w.

497. If p varies inversely with q and $p = 5$ when $q = 6$, find the equation that relates p and q.

498. If y varies inversely with x and $y = 11$ when $x = 3$ find the equation that relates x and y.

499. If p varies directly as q and $p = 10,$ when $q = 2.$ find the equation that relates p and q.

500. If v varies inversely with w and $v = 18$ when $w = \frac{1}{3}$ find the equation that relates v and w.

501. The force needed to break a board varies inversely with its length. If Tom uses 20 pounds of pressure to break a 1.5-foot long board, how many pounds of pressure would he need to use to break a 6 foot long board?

502. The number of hours it takes for ice to melt varies inversely with the air temperature. A block of ice melts in 2.5 hours when the temperature is 54 degrees. How long would it take for the same block of ice to melt if the temperature was 45 degrees?

503. The length a spring stretches varies directly with a weight placed at the end of the spring. When Meredith placed a 6-pound cantaloupe on a hanging scale, the spring stretched 2 inches. How far would the spring stretch if the cantaloupe weighed 9 pounds?

504. The amount that June gets paid varies directly the number of hours she works. When she worked 15 hours, she got paid $111. How much will she be paid for working 18 hours?

505. The fuel consumption (mpg) of a car varies inversely with its weight. A Ford Focus weighs 3000 pounds and gets 28.7 mpg on the highway. What would the fuel consumption be for a Ford Expedition that weighs 5,500 pounds? Round to the nearest tenth.

506. The volume of a gas in a container varies inversely as the pressure on the gas. If a container of argon has a volume of 336 cubic inches under a pressure of 2,500 psi, what will be its volume if the pressure is decreased to 2,000 psi?

507. The distance an object falls varies directly to the square of the time it falls. If an object falls 52.8 feet in 4 seconds, how far will it fall in 9 seconds?

508. The area of the face of a Ferris wheel varies directly with the square of its radius. If the area of one face of a Ferris wheel with diameter 150 feet is 70,650 square feet, what is the area of one face of a Ferris wheel with diameter of 16 feet?

Everyday Math

509. Ride Service It costs $35 for a ride from the city center to the airport, 14 miles away.

ⓐ Write the equation that relates the cost, *c*, with the number of miles, *m*.

ⓑ What would it cost to travel 22 miles with this service?

510. Road Trip The number of hours it takes Jack to drive from Boston to Bangor is inversely proportional to his average driving speed. When he drives at an average speed of 40 miles per hour, it takes him 6 hours for the trip.

ⓐ Write the equation that relates the number of hours, *h*, with the speed, *s*.

ⓑ How long would the trip take if his average speed was 75 miles per hour?

Writing Exercises

511. In your own words, explain the difference between direct variation and inverse variation.

512. Make up an example from your life experience of inverse variation.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
solve direct variation problems.			
solve inverse variation problems.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next chapter? Why or why not?*

CHAPTER 8 REVIEW

KEY TERMS

complex rational expression A complex rational expression is a rational expression in which the numerator or denominator contains a rational expression.

extraneous solution to a rational equation An extraneous solution to a rational equation is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

proportion A proportion is an equation of the form $\frac{a}{b}=\frac{c}{d}$, where $b \neq 0,\ d \neq 0.$ The proportion is read " a is to b, as c is to $d.$"

rational equation A rational equation is two rational expressions connected by an equal sign.

rational expression A rational expression is an expression of the form $\frac{p}{q}$, where *p* and *q* are polynomials and $q \neq 0.$

similar figures Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides are in the same ratio.

KEY CONCEPTS

8.1 Simplify Rational Expressions

- **Determine the Values for Which a Rational Expression is Undefined**

  Step 1. Set the denominator equal to zero.

  Step 2. Solve the equation, if possible.

- **Simplified Rational Expression**
  - A rational expression is considered simplified if there are no common factors in its numerator and denominator.

- **Simplify a Rational Expression**

  Step 1. Factor the numerator and denominator completely.

  Step 2. Simplify by dividing out common factors.

- **Opposites in a Rational Expression**
  - The opposite of $a-b$ is $b-a$.

    $\frac{a-b}{b-a}=-1 \qquad a \neq 0,\ b \neq 0,\ \mathrm{a} \neq \mathrm{b}$

8.2 Multiply and Divide Rational Expressions

- **Multiplication of Rational Expressions**
  - If $p,\ q,\ r,\ s$ are polynomials where $q \neq 0,\ s \neq 0$, then $\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}$.
  - To multiply rational expressions, multiply the numerators and multiply the denominators

- **Multiply a Rational Expression**

  Step 1. Factor each numerator and denominator completely.

  Step 2. Multiply the numerators and denominators.

  Step 3. Simplify by dividing out common factors.

- **Division of Rational Expressions**
  - If $p,\ q,\ r,\ s$ are polynomials where $q \neq 0,\ r \neq 0,\ s \neq 0$, then $\frac{p}{q} \div \frac{r}{s}=\frac{p}{q}\cdot\frac{s}{r}$.
  - To divide rational expressions multiply the first fraction by the reciprocal of the second.

- **Divide Rational Expressions**

  Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.

  Step 2. Factor the numerators and denominators completely.

Step 3. Multiply the numerators and denominators together.

Step 4. Simplify by dividing out common factors.

8.3 Add and Subtract Rational Expressions with a Common Denominator

- **Rational Expression Addition**
  - If $p, q,$ and r are polynomials where $r \neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r}$$

  - To add rational expressions with a common denominator, add the numerators and place the sum over the common denominator.
- **Rational Expression Subtraction**
  - If $p, q,$ and r are polynomials where $r \neq 0$, then

$$\frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

  - To subtract rational expressions, subtract the numerators and place the difference over the common denominator.

8.4 Add and Subtract Rational Expressions with Unlike Denominators

- **Find the Least Common Denominator of Rational Expressions**

Step 1. Factor each expression completely.

Step 2. List the factors of each expression. Match factors vertically when possible.

Step 3. Bring down the columns.

Step 4. Multiply the factors.

- **Add or Subtract Rational Expressions**

Step 1. Determine if the expressions have a common denominator.
Yes – go to step 2.
No – Rewrite each rational expression with the LCD.
  - Find the LCD.
  - Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 2. Add or subtract the rational expressions.

Step 3. Simplify, if possible.

8.5 Simplify Complex Rational Expressions

- **To Simplify a Rational Expression by Writing it as Division**

Step 1. Simplify the numerator and denominator.

Step 2. Rewrite the complex rational expression as a division problem.

Step 3. Divide the expressions.

- **To Simplify a Complex Rational Expression by Using the LCD**

Step 1. Find the LCD of all fractions in the complex rational expression.

Step 2. Multiply the numerator and denominator by the LCD.

Step 3. Simplify the expression.

8.6 Solve Rational Equations

- **Strategy to Solve Equations with Rational Expressions**

Step 1. Note any value of the variable that would make any denominator zero.

Step 2. Find the least common denominator of *all* denominators in the equation.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.

Step 4. Solve the resulting equation.

Step 5. Check.
- If any values found in Step 1 are algebraic solutions, discard them.
- Check any remaining solutions in the original equation.

8.7 Solve Proportion and Similar Figure Applications

- **Property of Similar Triangles**
  - If ΔABC is similar to ΔXYZ, then their corresponding angle measures are equal and their corresponding sides are in the same ratio.
- **Problem Solving Strategy for Geometry Applications**

Step 1. **Read** the problem and make sure all the words and ideas are understood. Draw the figure and label it with the given information.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for by choosing a variable to represent it.

Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

Step 5. **Solve the equation** using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

REVIEW EXERCISES

8.1 Simplify Rational Expressions

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

513. $\dfrac{2a+1}{3a-2}$

514. $\dfrac{b-3}{b^2-16}$

515. $\dfrac{3xy^2}{5y}$

516. $\dfrac{u-3}{u^2-u-30}$

Evaluate Rational Expressions

In the following exercises, evaluate the rational expressions for the given values.

517. $\dfrac{4p-1}{p^2+5}$ when $p=-1$

518. $\dfrac{q^2-5}{q+3}$ when q $=7$

519. $\dfrac{y^2-8}{y^2-y-2}$ when $y=1$

520. $\dfrac{z^2+2}{4z-z^2}$ when z $=3$

Simplify Rational Expressions

In the following exercises, simplify.

521. $\dfrac{10}{24}$

522. $\dfrac{8m^4}{16mn^3}$

523. $\dfrac{14a-14}{a-1}$

524. $\dfrac{b^2+7b+12}{b^2+8b+16}$

Simplify Rational Expressions with Opposite Factors

In the following exercises, simplify.

525. $\frac{c^2 - c - 2}{4 - c^2}$

526. $\frac{d - 16}{16 - d}$

527. $\frac{7v - 35}{25 - v^2}$

528. $\frac{w^2 - 3w - 28}{49 - w^2}$

8.2 Multiply and Divide Rational Expressions

Multiply Rational Expressions

In the following exercises, multiply.

529. $\frac{3}{8} \cdot \frac{2}{15}$

530. $\frac{2xy^2}{8y^3} \cdot \frac{16y}{24x}$

531. $\frac{3a^2 + 21a}{a^2 + 6a - 7} \cdot \frac{a - 1}{ab}$

532. $\frac{5z^2}{5z^2 + 40z + 35} \cdot \frac{z^2 - 1}{3z}$

Divide Rational Expressions

In the following exercises, divide.

533. $\frac{t^2 - 4t + 12}{t^2 + 8t + 12} \div \frac{t^2 - 36}{6t}$

534. $\frac{r^2 - 16}{4} \div \frac{r^3 - 64}{2r^2 - 8r + 32}$

535. $\frac{11 + w}{w - 9} \div \frac{121 - w^2}{9 - w}$

536. $\frac{3y^2 - 12y - 63}{4y + 3} \div (6y^2 - 42y)$

537. $\frac{\frac{c^2 - 64}{3c^2 + 26c + 16}}{\frac{c^2 - 4c - 32}{15c + 10}}$

538. $\frac{8m^2 - 8m}{m - 4} \cdot \frac{m^2 + 2m - 24}{m^2 + 7m + 10} \div \frac{2m^2 - 6m}{m + 5}$

8.3 Add and Subtract Rational Expressions with a Common Denominator

Add Rational Expressions with a Common Denominator

In the following exercises, add.

539. $\frac{3}{5} + \frac{2}{5}$

540. $\frac{4a^2}{2a - 1} - \frac{1}{2a - 1}$

541. $\frac{p^2 + 10p}{p + 5} + \frac{25}{p + 5}$

542. $\frac{3x}{x - 1} + \frac{2}{x - 1}$

Subtract Rational Expressions with a Common Denominator

In the following exercises, subtract.

543. $\frac{d^2}{d + 4} - \frac{3d + 28}{d + 4}$

544. $\frac{z^2}{z + 10} - \frac{100}{z + 10}$

545. $\frac{4q^2 - q + 3}{q^2 + 6q + 5} - \frac{3q^2 - q - 6}{q^2 + 6q + 5}$

546. $\frac{5t + 4t + 3}{t^2 - 25} - \frac{4t^2 - 8t - 32}{t^2 - 25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add and subtract.

547. $\frac{18w}{6w-1}+\frac{3w-2}{1-6w}$

548. $\frac{a^2+3a}{a^2-4}-\frac{3a-8}{4-a^2}$

549. $\frac{2b^2+3b-15}{b^2-49}-\frac{b^2+16b-1}{49-b^2}$

550. $\frac{8y^2-10y+7}{2y-5}+\frac{2y^2+7y+2}{5-2y}$

8.4 Add and Subtract Rational Expressions With Unlike Denominators

Find the Least Common Denominator of Rational Expressions

In the following exercises, find the LCD.

551. $\frac{4}{m^2-3m-10}, \frac{2m}{m^2-m-20}$

552. $\frac{6}{n^2-4}, \frac{2n}{n^2-4n+4}$

553. $\frac{5}{3p^2+17p-6}, \frac{2m}{3p^2-23p-8}$

Find Equivalent Rational Expressions

In the following exercises, rewrite as equivalent rational expressions with the given denominator.

554. Rewrite as equivalent rational expressions with denominator $(m+2)(m-5)(m+4)$: $\frac{4}{m^2-3m-10}, \frac{2m}{m^2-m-20}$.

555. Rewrite as equivalent rational expressions with denominator $(n-2)(n-2)(n+2)$: $\frac{6}{n^2-4n+4}, \frac{2n}{n^2-4}$.

556. Rewrite as equivalent rational expressions with denominator $(3p+1)(p+6)(p+8)$: $\frac{5}{3p^2+19p+6}, \frac{7p}{3p^2+25p+8}$

Add Rational Expressions with Different Denominators

In the following exercises, add.

557. $\frac{2}{3}+\frac{3}{5}$

558. $\frac{7}{5a}+\frac{3}{2b}$

559. $\frac{2}{c-2}+\frac{9}{c+3}$

560. $\frac{3d}{d^2-9}+\frac{5}{d^2+6d+9}$

561. $\frac{2x}{x^2+10x+24}+\frac{3x}{x^2+8x+16}$

562. $\frac{5q}{p^2q-p^2}+\frac{4q}{q^2-1}$

Subtract Rational Expressions with Different Denominators

In the following exercises, subtract and add.

563. $\frac{3v}{v+2}-\frac{v+2}{v+8}$

564. $\frac{-3w-15}{w^2+w-20}-\frac{w+2}{4-w}$

565. $\frac{7m+3}{m+2}-5$

566. $\frac{n}{n+3}+\frac{2}{n-3}-\frac{n-9}{n^2-9}$

567. $\frac{8d}{d^2-64}-\frac{4}{d+8}$

568. $\frac{5}{12x^2y}+\frac{7}{20xy^3}$

8.5 Simplify Complex Rational Expressions

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify.

569. $\dfrac{\frac{5a}{a+2}}{\frac{10a^2}{a^2-4}}$

570. $\dfrac{\frac{2}{5}+\frac{5}{6}}{\frac{1}{3}+\frac{1}{4}}$

571. $\dfrac{x-\frac{3x}{x+5}}{\frac{1}{x+5}+\frac{1}{x-5}}$

572. $\dfrac{\frac{2}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

573. $\dfrac{6+\frac{2}{q-4}}{\frac{5}{q+4}}$

574. $\dfrac{\frac{3}{a^2}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b^2}}$

575. $\dfrac{\frac{2}{z^2-49}+\frac{1}{z+7}}{\frac{9}{z+7}+\frac{12}{z-7}}$

576. $\dfrac{\frac{3}{y^2-4y-32}}{\frac{2}{y-8}+\frac{1}{y+4}}$

8.6 Solve Rational Equations

Solve Rational Equations

In the following exercises, solve.

577. $\frac{1}{2}+\frac{2}{3}=\frac{1}{x}$

578. $1-\frac{2}{m}=\frac{8}{m^2}$

579. $\frac{1}{b-2}+\frac{1}{b+2}=\frac{3}{b^2-4}$

580. $\frac{3}{q+8}-\frac{2}{q-2}=1$

581. $\frac{v-15}{v^2-9v+18}=\frac{4}{v-3}+\frac{2}{v-6}$

582. $\frac{z}{12}+\frac{z+3}{3z}=\frac{1}{z}$

Solve a Rational Equation for a Specific Variable

In the following exercises, solve for the indicated variable.

583. $\frac{V}{l}=hw$ for l

584. $\frac{1}{x}-\frac{2}{y}=5$ for y

585. $x=\frac{y+5}{z-7}$ for z

586. $P=\frac{k}{V}$ for V

8.7 Solve Proportion and Similar Figure Applications Similarity

Solve Proportions

In the following exercises, solve.

587. $\frac{x}{4}=\frac{3}{5}$

588. $\frac{3}{y}=\frac{9}{5}$

589. $\frac{s}{s+20}=\frac{3}{7}$

590. $\frac{t-3}{5}=\frac{t+2}{9}$

In the following exercises, solve using proportions.

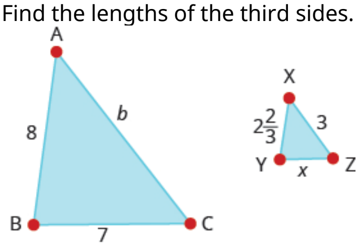
591. Rachael had a 21 ounce strawberry shake that has 739 calories. How many calories are there in a 32 ounce shake?

592. Leo went to Mexico over Christmas break and changed $525 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 16.25 Mexican pesos. How many Mexican pesos did he get for his trip?

Solve Similar Figure Applications

In the following exercises, solve.

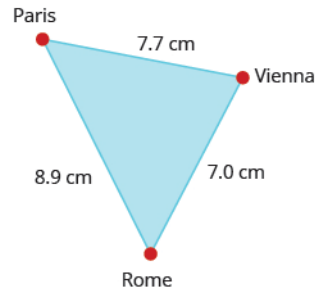
593. ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure. Find the lengths of the third sides.



594. On a map of Europe, Paris, Rome, and Vienna form a triangle whose sides are shown in the figure below. If the actual distance from Rome to Vienna is 700 miles, find the distance from

ⓐ Paris to Rome

ⓑ Paris to Vienna



595. Tony is 5.75 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a nearby tree was 32 feet long. Find the height of the tree.

596. The height of a lighthouse in Pensacola, Florida is 150 feet. Standing next to the statue, 5.5 foot tall Natalie cast a 1.1 foot shadow How long would the shadow of the lighthouse be?

8.8 Solve Uniform Motion and Work Applications Problems

Solve Uniform Motion Applications

In the following exercises, solve.

597. When making the 5-hour drive home from visiting her parents, Lisa ran into bad weather. She was able to drive 176 miles while the weather was good, but then driving 10 mph slower, went 81 miles in the bad weather. How fast did she drive when the weather was bad?

598. Mark is riding on a plane that can fly 490 miles with a tailwind of 20 mph in the same time that it can fly 350 miles against a tailwind of 20 mph. What is the speed of the plane?

599. John can ride his bicycle 8 mph faster than Luke can ride his bike. It takes Luke 3 hours longer than John to ride 48 miles. How fast can John ride his bike?

600. Mark was training for a triathlon. He ran 8 kilometers and biked 32 kilometers in a total of 3 hours. His running speed was 8 kilometers per hour less than his biking speed. What was his running speed?

Solve Work Applications

In the following exercises, solve.

601. Jerry can frame a room in 1 hour, while Jake takes 4 hours. How long could they frame a room working together?

602. Lisa takes 3 hours to mow the lawn while her cousin, Barb, takes 2 hours. How long will it take them working together?

603. Jeffrey can paint a house in 6 days, but if he gets a helper he can do it in 4 days. How long would it take the helper to paint the house alone?

604. Sue and Deb work together writing a book that takes them 90 days. If Sue worked alone it would take her 120 days. How long would it take Deb to write the book alone?

8.9 Use Direct and Inverse Variation

Solve Direct Variation Problems

In the following exercises, solve.

605. If y varies directly as x, when $y = 9$ and $x = 3$, find x when $y = 21.$

606. If y varies inversely as x, when $y = 20$ and $x = 2$ find y when $x = 4.$

607. If m varies inversely with the square of n, when $m = 4$ and $n = 6$ find m when $n = 2.$

608. Vanessa is traveling to see her fiancé. The distance, *d*, varies directly with the speed, *v*, she drives. If she travels 258 miles driving 60 mph, how far would she travel going 70 mph?

609. If the cost of a pizza varies directly with its diameter, and if an 8” diameter pizza costs $12, how much would a 6” diameter pizza cost?

610. The distance to stop a car varies directly with the square of its speed. It takes 200 feet to stop a car going 50 mph. How many feet would it take to stop a car going 60 mph?

Solve Inverse Variation Problems

In the following exercises, solve.

611. The number of tickets for a music fundraiser varies inversely with the price of the tickets. If Madelyn has just enough money to purchase 12 tickets for $6, how many tickets can Madelyn afford to buy if the price increased to $8?

612. On a string instrument, the length of a string varies inversely with the frequency of its vibrations. If an 11-inch string on a violin has a frequency of 360 cycles per second, what frequency does a 12 inch string have?

PRACTICE TEST

In the following exercises, simplify.

613. $\dfrac{3a^2 b}{6ab^2}$

614. $\dfrac{5b-25}{b^2-25}$

In the following exercises, perform the indicated operation and simplify.

615. $\dfrac{4x}{x+2} \cdot \dfrac{x^2+5x+6}{12x^2}$

616. $\dfrac{5y}{4y-8} \cdot \dfrac{y^2-4}{10}$

617. $\dfrac{4}{pq} + \dfrac{5}{p}$

618. $\dfrac{1}{z-9} - \dfrac{3}{z+9}$

619. $\dfrac{\frac{2}{3}+\frac{3}{5}}{\frac{2}{5}}$

620. $\dfrac{\frac{1}{m}-\frac{1}{n}}{\frac{1}{n}+\frac{1}{m}}$

In the following exercises, solve each equation.

621. $\dfrac{1}{2} + \dfrac{2}{7} = \dfrac{1}{x}$

622. $\dfrac{5}{y-6} = \dfrac{3}{y+6}$

623. $\dfrac{1}{z-5} + \dfrac{1}{z+5} = \dfrac{1}{z^2-25}$

624. $\dfrac{t}{4} = \dfrac{3}{5}$

625. $\dfrac{2}{r-2} = \dfrac{3}{r-1}$

In the following exercises, solve.

626. If y varies directly with x, and $x = 5$ when $y = 30$, find x when $y = 42$.

627. If y varies inversely with x and $x = 6$ when $y = 20$, find y when $x = 2$.

628. If y varies inversely with the square of x and $x = 3$ when $y = 9$, find y when $x = 4$.

629. The recommended erythromycin dosage for dogs, is 5 mg for every pound the dog weighs. If Daisy weighs 25 pounds, how many milligrams of erythromycin should her veterinarian prescribe?

630. Julia spent 4 hours Sunday afternoon exercising at the gym. She ran on the treadmill for 10 miles and then biked for 20 miles. Her biking speed was 5 mph faster than her running speed on the treadmill. What was her running speed?

631. Kurt can ride his bike for 30 miles with the wind in the same amount of time that he can go 21 miles against the wind. If the wind's speed is 6 mph, what is Kurt's speed on his bike?

632. Amanda jogs to the park 8 miles using one route and then returns via a 14-mile route. The return trip takes her 1 hour longer than her jog to the park. Find her jogging rate.

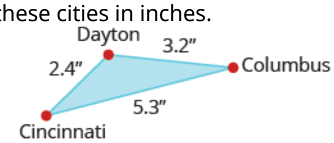
633. An experienced window washer can wash all the windows in Mike's house in 2 hours, while a new trainee can wash all the windows in 7 hours. How long would it take them working together?

634. Josh can split a truckload of logs in 8 hours, but working with his dad they can get it done in 3 hours. How long would it take Josh's dad working alone to split the logs?

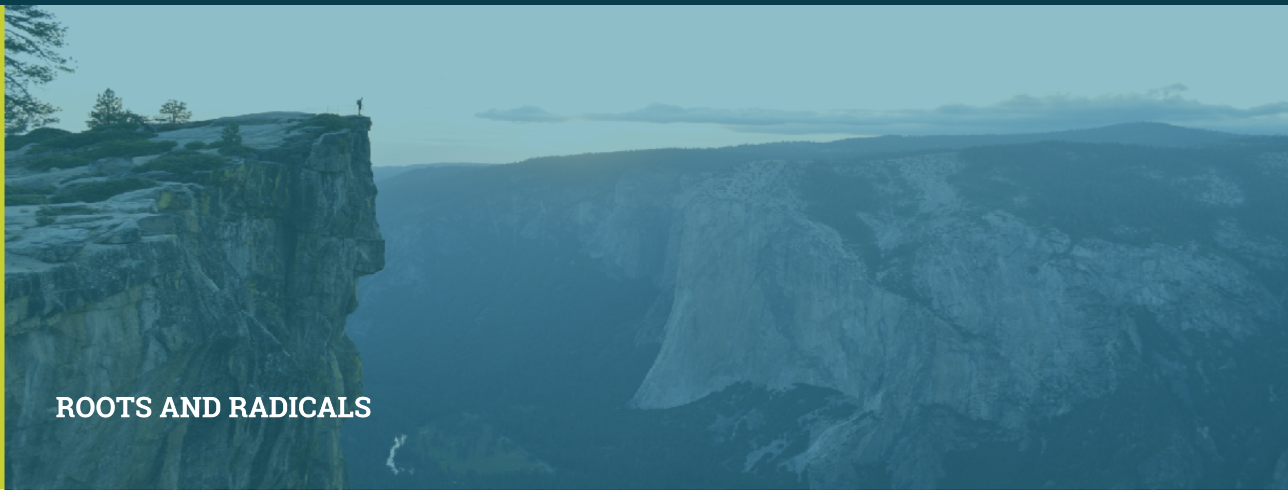
635. The price that Tyler pays for gas varies directly with the number of gallons he buys. If 24 gallons cost him $59.76, what would 30 gallons cost?

636. The volume of a gas in a container varies inversely with the pressure on the gas. If a container of nitrogen has a volume of 29.5 liters with 2000 psi, what is the volume if the tank has a 14.7 psi rating? Round to the nearest whole number.

637. The cities of Dayton, Columbus, and Cincinnati form a triangle in southern Ohio, as shown on the figure below, that gives the map distances between these cities in inches.



The actual distance from Dayton to Cincinnati is 48 miles. What is the actual distance between Dayton and Columbus?



9

ROOTS AND RADICALS

Figure 9.1 Square roots are used to determine the time it would take for a stone falling from the edge of this cliff to hit the land below.

Chapter Outline

Introduction

Suppose a stone falls from the edge of a cliff. The number of feet the stone has dropped after t seconds can be found by multiplying 16 times the square of t. But to calculate the number of seconds it would take the stone to hit the land below, we need to use a square root. In this chapter, we will introduce and apply the properties of square roots, and extend these concepts to higher order roots and rational exponents.

9.1 Simplify and Use Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with square roots
- Estimate square roots
- Approximate square roots
- Simplify variable expressions with square roots

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: ⓐ 9^2 ⓑ $(-9)^2$ ⓒ -9^2.
   If you missed this problem, review Example 1.50.
2. Round 3.846 to the nearest hundredth.
   If you missed this problem, review Example 1.94.

3. For each number, identify whether it is a real number or not a real number:
ⓐ $-\sqrt{100}$ ⓑ $\sqrt{-100}$.
If you missed this problem, review **Example 1.113**.

Simplify Expressions with Square Roots

Remember that when a number n is multiplied by itself, we write n^2 and read it "n squared." For example, 15^2 reads as "15 squared," and 225 is called the square of 15, since $15^2 = 225$.

Square of a Number

If $n^2 = m$, then m is the square of n .

Sometimes we will need to look at the relationship between numbers and their squares in reverse. Because 225 is the square of 15, we can also say that 15 is a square root of 225. A number whose square is m is called a *square root* of m .

Square Root of a Number

If $n^2 = m$, then n is a square root of m .

Notice $(-15)^2 = 225$ also, so -15 is also a square root of 225. Therefore, both 15 and -15 are square roots of 225.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? The *radical sign,* $\sqrt{m}$, denotes the positive square root. The positive square root is also called the principal square root.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Square Root Notation

$\sqrt{m}$ is read as "the square root of m ."

If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

The square root of m , $\sqrt{m}$, is the positive number whose square is m .

Since 15 is the positive square root of 225, we write $\sqrt{225} = 15$. Fill in **Figure 9.2** to make a table of square roots you can refer to as you work this chapter.

$\sqrt{1}$	$\sqrt{4}$	$\sqrt{9}$	$\sqrt{16}$	$\sqrt{25}$	$\sqrt{36}$	$\sqrt{49}$	$\sqrt{64}$	$\sqrt{81}$	$\sqrt{100}$	$\sqrt{121}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{196}$	$\sqrt{225}$
														15

Figure 9.2

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{225} = 15$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{225} = -15$.

EXAMPLE 9.1

Simplify: ⓐ $\sqrt{36}$ ⓑ $\sqrt{196}$ ⓒ $-\sqrt{81}$ ⓓ $-\sqrt{289}$.

Solution

ⓐ

	$\sqrt{36}$
Since $6^2 = 36$	6

ⓑ

	$\sqrt{196}$
Since $14^2 = 196$	14

ⓒ

	$-\sqrt{81}$
The negative is in front of the radical sign.	-9

ⓓ

	$-\sqrt{289}$
The negative is in front of the radical sign.	-17

TRY IT : : 9.1 Simplify: ⓐ $-\sqrt{49}$ ⓑ $\sqrt{225}$.

TRY IT : : 9.2 Simplify: ⓐ $\sqrt{64}$ ⓑ $-\sqrt{121}$.

EXAMPLE 9.2

Simplify: ⓐ $\sqrt{-169}$ ⓑ $-\sqrt{64}$.

Solution

ⓐ

	$\sqrt{-169}$
There is no real number whose square is -169.	$\sqrt{-169}$ is not a real number.

ⓑ

	$-\sqrt{64}$
The negative is in front of the radical.	-8

TRY IT : : 9.3 Simplify: ⓐ $\sqrt{-196}$ ⓑ $-\sqrt{81}$.

TRY IT : : 9.4 Simplify: ⓐ $-\sqrt{49}$ ⓑ $\sqrt{-121}$.

When using the order of operations to simplify an expression that has square roots, we treat the radical as a grouping symbol.

EXAMPLE 9.3

Simplify: ⓐ $\sqrt{25} + \sqrt{144}$ ⓑ $\sqrt{25 + 144}$.

Solution

ⓐ

	$\sqrt{25} + \sqrt{144}$
Use the order of operations.	$5 + 12$
Simplify.	17

ⓑ

	$\sqrt{25 + 144}$
Simplify under the radical sign.	$\sqrt{169}$
Simplify.	13

Notice the different answers in parts ⓐ and ⓑ!

TRY IT : : 9.5 Simplify: ⓐ $\sqrt{9} + \sqrt{16}$ ⓑ $\sqrt{9 + 16}$.

TRY IT : : 9.6 Simplify: ⓐ $\sqrt{64 + 225}$ ⓑ $\sqrt{64} + \sqrt{225}$.

Estimate Square Roots

So far we have only considered square roots of perfect square numbers. The square roots of other numbers are not whole numbers. Look at Table 9.1 below.

Number	Square Root
4	$\sqrt{4} = 2$
5	$\sqrt{5}$
6	$\sqrt{6}$
7	$\sqrt{7}$
8	$\sqrt{8}$
9	$\sqrt{9} = 3$

Table 9.1

The square roots of numbers between 4 and 9 must be between the two consecutive whole numbers 2 and 3, and they are not whole numbers. Based on the pattern in the table above, we could say that $\sqrt{5}$ must be between 2 and 3. Using inequality symbols, we write:

$$2 < \sqrt{5} < 3$$

EXAMPLE 9.4

Estimate $\sqrt{60}$ between two consecutive whole numbers.

Solution

Think of the perfect square numbers closest to 60. Make a small table of these perfect squares and their squares roots.

Number	Square root
36	6
49	7
64	8
81	9

60 $\sqrt{60}$

Locate 60 between two consecutive perfect squares.	$49 < 60 < 64$
$\sqrt{60}$ is between their square roots.	$7 < \sqrt{60} < 8$

TRY IT : : 9.7 Estimate the square root $\sqrt{38}$ between two consecutive whole numbers.

TRY IT : : 9.8 Estimate the square root $\sqrt{84}$ between two consecutive whole numbers.

Approximate Square Roots

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find them. Find the $\sqrt{x}$ key on your calculator. You will use this key to approximate square roots.

When you use your calculator to find the square root of a number that is not a perfect square, the answer that you see is not the exact square root. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately.'

Suppose your calculator has a 10-digit display. You would see that

$$\sqrt{5} \approx 2.236067978$$

If we wanted to round $\sqrt{5}$ to two decimal places, we would say

$$\sqrt{5} \approx 2.24$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:

$$\begin{aligned} (2.236067978)^2 &= 5.000000002 \\ (2.24)^2 &= 5.0176 \end{aligned}$$

Their squares are close to 5, but are not exactly equal to 5.

Using the square root key on a calculator and then rounding to two decimal places, we can find:

$$\begin{aligned} \sqrt{4} &= 2 \\ \sqrt{5} &\approx 2.24 \\ \sqrt{6} &\approx 2.45 \\ \sqrt{7} &\approx 2.65 \\ \sqrt{8} &\approx 2.83 \\ \sqrt{9} &= 3 \end{aligned}$$

EXAMPLE 9.5

Round $\sqrt{17}$ to two decimal places.

Solution

	$\sqrt{17}$
Use the calculator square root key.	$4.123105626...$
Round to two decimal places.	4.12
	$\sqrt{17} \approx 4.12$

TRY IT : : 9.9 Round $\sqrt{11}$ to two decimal places.

TRY IT : : 9.10 Round $\sqrt{13}$ to two decimal places.

Simplify Variable Expressions with Square Roots

What if we have to find a square root of an expression with a variable? Consider $\sqrt{9x^2}$. Can you think of an expression whose square is $9x^2$?

$$\begin{aligned}(\,?\,)^2 &= 9x^2\\ (3x)^2 &= 9x^2, \qquad \text{so } \sqrt{9x^2} = 3x\end{aligned}$$

When we use the radical sign to take the square root of a variable expression, we should specify that $x \geq 0$ to make sure we get the *principal square root*.

However, in this chapter we will assume that each variable in a square-root expression represents a non-negative number and so we will not write $x \geq 0$ next to every radical.

What about square roots of higher powers of variables? Think about the Power Property of Exponents we used in Chapter 6.

$$(a^m)^n = a^{m \cdot n}$$

If we square a^m, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}$$

How does this help us take square roots? Let's look at a few:

$$\sqrt{25u^8} = 5u^4 \text{ because } \left(5u^4\right)^2 = 25u^8$$
$$\sqrt{16r^{20}} = 4r^{10} \text{ because } \left(4r^{10}\right)^2 = 16r^{20}$$
$$\sqrt{196q^{36}} = 14q^{18} \text{ because } \left(14q^{18}\right)^2 = 196q^{36}$$

EXAMPLE 9.6

Simplify: ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$.

Solution

ⓐ

	$\sqrt{x^6}$
Since $\left(x^3\right)^2 = x^6$.	x^3

ⓑ

	$\sqrt{y^{16}}$
Since $\left(y^8\right)^2 = y^{16}$.	y^8

TRY IT : : 9.11 Simplify: ⓐ $\sqrt{y^8}$ ⓑ $\sqrt{z^{12}}$.

TRY IT : : 9.12 Simplify: ⓐ $\sqrt{m^4}$ ⓑ $\sqrt{b^{10}}$.

EXAMPLE 9.7

Simplify: $\sqrt{16n^2}$.

Solution

	$\sqrt{16n^2}$
Since $(4n)^2 = 16n^2$.	$4n$

TRY IT : : 9.13 Simplify: $\sqrt{64x^2}$.

TRY IT : : 9.14 Simplify: $\sqrt{169y^2}$.

EXAMPLE 9.8

Simplify: $-\sqrt{81c^2}$.

Solution

	$-\sqrt{81c^2}$
Since $(9c)^2 = 81c^2$.	$-9c$

TRY IT : : 9.15 Simplify: $-\sqrt{121y^2}$.

TRY IT : : 9.16 Simplify: $-\sqrt{100p^2}$.

EXAMPLE 9.9

Simplify: $\sqrt{36x^2y^2}$.

Solution

	$\sqrt{36x^2y^2}$
Since $(6xy)^2 = 36x^2y^2$.	$6xy$

TRY IT : : 9.17 Simplify: $\sqrt{100a^2b^2}$.

TRY IT : : 9.18 Simplify: $\sqrt{225m^2n^2}$.

EXAMPLE 9.10

Simplify: $\sqrt{64p^{64}}$.

Solution

	$\sqrt{64p^{64}}$
Since $\left(8p^{32}\right)^2 = 64p^{64}$.	$8p^{32}$

TRY IT : : 9.19 Simplify: $\sqrt{49x^{30}}$.

TRY IT : : 9.20 Simplify: $\sqrt{81w^{36}}$.

EXAMPLE 9.11

Simplify: $\sqrt{121a^6 b^8}$

Solution

	$\sqrt{121a^6 b^8}$
Since $\left(11a^3 b^4\right)^2 = 121a^6 b^8$.	$11a^3 b^4$

TRY IT : : 9.21 Simplify: $\sqrt{169x^{10} y^{14}}$.

TRY IT : : 9.22 Simplify: $\sqrt{144p^{12} q^{20}}$.

MEDIA : :

Access this online resource for additional instruction and practice with square roots.

- **Square Roots (https://openstax.org/l/25SquareRoots)**

9.1 EXERCISES

Practice Makes Perfect

Simplify Expressions with Square Roots

In the following exercises, simplify.

1. $\sqrt{36}$

2. $\sqrt{4}$

3. $\sqrt{64}$

4. $\sqrt{169}$

5. $\sqrt{9}$

6. $\sqrt{16}$

7. $\sqrt{100}$

8. $\sqrt{144}$

9. $-\sqrt{4}$

10. $-\sqrt{100}$

11. $-\sqrt{1}$

12. $-\sqrt{121}$

13. $\sqrt{-121}$

14. $\sqrt{-36}$

15. $\sqrt{-9}$

16. $\sqrt{-49}$

17. $\sqrt{9+16}$

18. $\sqrt{25+144}$

19. $\sqrt{9}+\sqrt{16}$

20. $\sqrt{25}+\sqrt{144}$

Estimate Square Roots

In the following exercises, estimate each square root between two consecutive whole numbers.

21. $\sqrt{70}$

22. $\sqrt{55}$

23. $\sqrt{200}$

24. $\sqrt{172}$

Approximate Square Roots

In the following exercises, approximate each square root and round to two decimal places.

25. $\sqrt{19}$

26. $\sqrt{21}$

27. $\sqrt{53}$

28. $\sqrt{47}$

Simplify Variable Expressions with Square Roots

In the following exercises, simplify.

29. $\sqrt{y^2}$

30. $\sqrt{b^2}$

31. $\sqrt{a^{14}}$

32. $\sqrt{w^{24}}$

33. $\sqrt{49x^2}$

34. $\sqrt{100y^2}$

35. $\sqrt{121m^{20}}$

36. $\sqrt{25h^{44}}$

37. $\sqrt{81x^{36}}$

38. $\sqrt{144z^{84}}$

39. $-\sqrt{81x^{18}}$

40. $-\sqrt{100m^{32}}$

41. $-\sqrt{64a^2}$

42. $-\sqrt{25x^2}$

43. $\sqrt{144x^2y^2}$

44. $\sqrt{196a^2b^2}$

45. $\sqrt{169w^8y^{10}}$

46. $\sqrt{81p^{24}q^6}$

47. $\sqrt{9c^8 d^{12}}$

48. $\sqrt{36r^6 s^{20}}$

Everyday Math

49. Decorating Denise wants to have a square accent of designer tiles in her new shower. She can afford to buy 625 square centimeters of the designer tiles. How long can a side of the accent be?

50. Decorating Morris wants to have a square mosaic inlaid in his new patio. His budget allows for 2025 square inch tiles. How long can a side of the mosaic be?

Writing Exercises

51. Why is there no real number equal to $\sqrt{-64}$?

52. What is the difference between 9^2 and $\sqrt{9}$?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with square roots.			
estimate square roots.			
approximate square roots.			
simplify variable expressions with square roots.			

ⓑ *On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

9.2 Simplify Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Use the Product Property to simplify square roots
- Use the Quotient Property to simplify square roots

Be Prepared!

Before you get started take this readiness quiz.

1. Simplify: $\frac{80}{176}$.
   If you missed this problem, review Example 1.65.
2. Simplify: $\frac{n^9}{n^3}$.
   If you missed this problem, review Example 6.59.
3. Simplify: $\frac{q^4}{q^{12}}$.
   If you missed this problem, review Example 6.60.

In the last section, we estimated the square root of a number between two consecutive whole numbers. We can say that $\sqrt{50}$ is between 7 and 8. This is fairly easy to do when the numbers are small enough that we can use Figure 9.2.

But what if we want to estimate $\sqrt{500}$? If we simplify the square root first, we'll be able to estimate it easily. There are other reasons, too, to simplify square roots as you'll see later in this chapter.

A square root is considered *simplified* if its radicand contains no perfect square factors.

Simplified Square Root

$\sqrt{a}$ is considered simplified if a has no perfect square factors.

So $\sqrt{31}$ is simplified. But $\sqrt{32}$ is not simplified, because 16 is a perfect square factor of 32.

Use the Product Property to Simplify Square Roots

The properties we will use to simplify expressions with square roots are similar to the properties of exponents. We know that $(ab)^m = a^m b^m$. The corresponding property of square roots says that $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

Product Property of Square Roots

If *a*, *b* are non-negative real numbers, then $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

We use the Product Property of Square Roots to remove all perfect square factors from a radical. We will show how to do this in Example 9.12.

EXAMPLE 9.12 HOW TO USE THE PRODUCT PROPERTY TO SIMPLIFY A SQUARE ROOT

Simplify: $\sqrt{50}$.

Solution

Step 1. Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect square factor.	25 is the largest perfect square factor of 50. $50 = 25 \cdot 2$ Always write the perfect square factor first.	$\sqrt{50}$ $\sqrt{25 \cdot 2}$

Step 2. Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{25} \cdot \sqrt{2}$
Step 3. Simplify the square root of the perfect square.		$5\sqrt{2}$

TRY IT : : 9.23 Simplify: $\sqrt{48}$.

TRY IT : : 9.24 Simplify: $\sqrt{45}$.

Notice in the previous example that the simplified form of $\sqrt{50}$ is $5\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

HOW TO : : SIMPLIFY A SQUARE ROOT USING THE PRODUCT PROPERTY.

Step 1. Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect-square factor.

Step 2. Use the product rule to rewrite the radical as the product of two radicals.

Step 3. Simplify the square root of the perfect square.

EXAMPLE 9.13

Simplify: $\sqrt{500}$.

Solution

	$\sqrt{500}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{100 \cdot 5}$
Rewrite the radical as the product of two radicals.	$\sqrt{100} \cdot \sqrt{5}$
Simplify.	$10\sqrt{5}$

TRY IT : : 9.25 Simplify: $\sqrt{288}$.

TRY IT : : 9.26 Simplify: $\sqrt{432}$.

We could use the simplified form $10\sqrt{5}$ to estimate $\sqrt{500}$. We know 5 is between 2 and 3, and $\sqrt{500}$ is $10\sqrt{5}$. So $\sqrt{500}$ is between 20 and 30.

The next example is much like the previous examples, but with variables.

EXAMPLE 9.14

Simplify: $\sqrt{x^3}$.

Solution

	$\sqrt{x^3}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{x^2 \cdot x}$
Rewrite the radical as the product of two radicals.	$\sqrt{x^2} \cdot \sqrt{x}$
Simplify.	$x\sqrt{x}$

TRY IT : : 9.27 Simplify: $\sqrt{b^5}$.

TRY IT : : 9.28 Simplify: $\sqrt{p^9}$.

We follow the same procedure when there is a coefficient in the radical, too.

EXAMPLE 9.15

Simplify: $\sqrt{25y^5}$.

Solution

	$\sqrt{25y^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{25y^4 \cdot y}$
Rewrite the radical as the product of two radicals.	$\sqrt{25y^4} \cdot \sqrt{y}$
Simplify.	$5y^2\sqrt{y}$

TRY IT : : 9.29 Simplify: $\sqrt{16x^7}$.

TRY IT : : 9.30 Simplify: $\sqrt{49v^9}$.

In the next example both the constant and the variable have perfect square factors.

EXAMPLE 9.16

Simplify: $\sqrt{72n^7}$.

Solution

	$\sqrt{72n^7}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{36n^6 \cdot 2n}$
Rewrite the radical as the product of two radicals.	$\sqrt{36n^6} \cdot \sqrt{2n}$
Simplify.	$6n^3\sqrt{2n}$

TRY IT : : 9.31 Simplify: $\sqrt{32y^5}$.

TRY IT : : 9.32 Simplify: $\sqrt{75a^9}$.

EXAMPLE 9.17

Simplify: $\sqrt{63u^3v^5}$.

Solution

	$\sqrt{63u^3v^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9u^2v^4 \cdot 7uv}$
Rewrite the radical as the product of two radicals.	$\sqrt{9u^2v^4} \cdot \sqrt{7uv}$
Simplify.	$3uv^2\sqrt{7uv}$

TRY IT : : 9.33 Simplify: $\sqrt{98a^7b^5}$.

TRY IT : : 9.34 Simplify: $\sqrt{180m^9n^{11}}$.

We have seen how to use the Order of Operations to simplify some expressions with radicals. To simplify $\sqrt{25} + \sqrt{144}$ we must simplify each square root separately first, then add to get the sum of 17.

The expression $\sqrt{17} + \sqrt{7}$ cannot be simplified—to begin we'd need to simplify each square root, but neither 17 nor 7 contains a perfect square factor.

In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer.

EXAMPLE 9.18

Simplify: $3 + \sqrt{32}$.

Solution

	$3 + \sqrt{32}$
Rewrite the radicand as a product using the largest perfect square factor.	$3 + \sqrt{16 \cdot 2}$
Rewrite the radical as the product of two radicals.	$3 + \sqrt{16} \cdot \sqrt{2}$
Simplify.	$3 + 4\sqrt{2}$

The terms are not like and so we cannot add them. Trying to add an integer and a radical is like trying to add an integer and a variable—they are not like terms!

TRY IT : : 9.35 Simplify: $5 + \sqrt{75}$.

TRY IT : : 9.36 Simplify: $2 + \sqrt{98}$.

The next example includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

EXAMPLE 9.19

Simplify: $\dfrac{4-\sqrt{48}}{2}$.

Solution

	$\dfrac{4-\sqrt{48}}{2}$
Rewrite the radicand as a product using the largest perfect square factor.	$\dfrac{4-\sqrt{16\cdot 3}}{2}$
Rewrite the radical as the product of two radicals.	$\dfrac{4-\sqrt{16}\cdot\sqrt{3}}{2}$
Simplify.	$\dfrac{4-4\sqrt{3}}{2}$
Factor the common factor from the numerator.	$\dfrac{4(1-\sqrt{3})}{2}$
Remove the common factor, 2, from the numerator and denominator.	$\dfrac{\cancel{2}\cdot 2(1-\sqrt{3})}{\cancel{2}}$
Simplify.	$2(1-\sqrt{3})$

TRY IT : : 9.37 Simplify: $\dfrac{10-\sqrt{75}}{5}$.

TRY IT : : 9.38 Simplify: $\dfrac{6-\sqrt{45}}{3}$.

Use the Quotient Property to Simplify Square Roots

Whenever you have to simplify a square root, the first step you should take is to determine whether the radicand is a perfect square. A *perfect square fraction* is a fraction in which both the numerator and the denominator are perfect squares.

EXAMPLE 9.20

Simplify: $\sqrt{\dfrac{9}{64}}$.

Solution

	$\sqrt{\dfrac{9}{64}}$
Since $\left(\dfrac{3}{8}\right)^2=\dfrac{9}{64}$	$\dfrac{3}{8}$

TRY IT : : 9.39 Simplify: $\sqrt{\dfrac{25}{16}}$.

TRY IT : : 9.40 Simplify: $\sqrt{\dfrac{49}{81}}$.

If the numerator and denominator have any common factors, remove them. You may find a perfect square fraction!

EXAMPLE 9.21

Simplify: $\sqrt{\frac{45}{80}}$.

Solution

	$\sqrt{\frac{45}{80}}$
Simplify inside the radical fir t. Rewrite showing the common factors of the numerator and denominator.	$\sqrt{\frac{5 \cdot 9}{5 \cdot 16}}$
Simplify the fraction by removing common factors.	$\sqrt{\frac{9}{16}}$
Simplify. $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$	$\frac{3}{4}$

TRY IT : : 9.41 Simplify: $\sqrt{\frac{75}{48}}$.

TRY IT : : 9.42 Simplify: $\sqrt{\frac{98}{162}}$.

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents, $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$.

EXAMPLE 9.22

Simplify: $\sqrt{\frac{m^6}{m^4}}$.

Solution

	$\sqrt{\frac{m^6}{m^4}}$
Simplify the fraction inside the radical fir t. Divide the like bases by subtracting the exponents.	$\sqrt{m^2}$
Simplify.	m

TRY IT : : 9.43 Simplify: $\sqrt{\frac{a^8}{a^6}}$.

TRY IT : : 9.44 Simplify: $\sqrt{\frac{x^{14}}{x^{10}}}$.

EXAMPLE 9.23

Simplify: $\sqrt{\dfrac{48p^7}{3p^3}}$.

Solution

	$\sqrt{\dfrac{48p^7}{3p^3}}$
Simplify the fraction inside the radical fir t.	$\sqrt{16p^4}$
Simplify.	$4p^2$

TRY IT : : 9.45 Simplify: $\sqrt{\dfrac{75x^5}{3x}}$.

TRY IT : : 9.46 Simplify: $\sqrt{\dfrac{72z^{12}}{2z^{10}}}$.

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$$

We can use a similar property to simplify a square root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect square we simplify the numerator and denominator separately.

Quotient Property of Square Roots

If *a*, *b* are non-negative real numbers and $b \neq 0$, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

EXAMPLE 9.24

Simplify: $\sqrt{\dfrac{21}{64}}$.

Solution

	$\sqrt{\dfrac{21}{64}}$
We cannot simplify the fraction inside the radical. Rewrite using the quotient property.	$\dfrac{\sqrt{21}}{\sqrt{64}}$
Simplify the square root of 64. The numerator cannot be simplified	$\dfrac{\sqrt{21}}{8}$

TRY IT : : 9.47 Simplify: $\sqrt{\dfrac{19}{49}}$.

TRY IT : : 9.48 Simplify: $\sqrt{\dfrac{28}{81}}$.

EXAMPLE 9.25 HOW TO USE THE QUOTIENT PROPERTY TO SIMPLIFY A SQUARE ROOT

Simplify: $\sqrt{\frac{27m^3}{196}}$.

Solution

Step 1. Simplify the fraction in the radicand, if possible.	$\frac{27m^3}{196}$ cannot be simplified.	$\sqrt{\frac{27m^3}{196}}$
Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.	We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.	$\frac{\sqrt{27m^3}}{\sqrt{196}}$
Step 3. Simplify the radicals in the numerator and the denominator.	$9m^2$ and 196 are perfect squares.	$\frac{\sqrt{9m^2}\cdot\sqrt{3m}}{\sqrt{196}}$ $\frac{3m\sqrt{3m}}{14}$

TRY IT : : 9.49 Simplify: $\sqrt{\frac{24p^3}{49}}$.

TRY IT : : 9.50 Simplify: $\sqrt{\frac{48x^5}{100}}$.

HOW TO : : SIMPLIFY A SQUARE ROOT USING THE QUOTIENT PROPERTY.

Step 1. Simplify the fraction in the radicand, if possible.
Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
Step 3. Simplify the radicals in the numerator and the denominator.

EXAMPLE 9.26

Simplify: $\sqrt{\frac{45x^5}{y^4}}$.

Solution

	$\sqrt{\frac{45x^5}{y^4}}$
We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.	$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{9x^4}\cdot\sqrt{5x}}{y^2}$
Simplify.	$\frac{3x^2\sqrt{5x}}{y^2}$

TRY IT : : 9.51 Simplify: $\sqrt{\dfrac{80m^3}{n^6}}$.

TRY IT : : 9.52 Simplify: $\sqrt{\dfrac{54u^7}{v^8}}$.

Be sure to simplify the fraction in the radicand first, if possible.

EXAMPLE 9.27

Simplify: $\sqrt{\dfrac{81d^9}{25d^4}}$.

Solution

	$\sqrt{\dfrac{81d^9}{25d^4}}$
Simplify the fraction in the radicand.	$\sqrt{\dfrac{81d^5}{25}}$
Rewrite using the Quotient Property.	$\dfrac{\sqrt{81d^5}}{\sqrt{25}}$
Simplify the radicals in the numerator and the denominator.	$\dfrac{\sqrt{81d^4}\cdot\sqrt{d}}{5}$
Simplify.	$\dfrac{9d^2\sqrt{d}}{5}$

TRY IT : : 9.53 Simplify: $\sqrt{\dfrac{64x^7}{9x^3}}$.

TRY IT : : 9.54 Simplify: $\sqrt{\dfrac{16a^9}{100a^5}}$.

EXAMPLE 9.28

Simplify: $\sqrt{\dfrac{18p^5q^7}{32pq^2}}$.

Solution

	$\sqrt{\dfrac{18p^5q^7}{32pq^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt{\dfrac{9p^4q^5}{16}}$
Rewrite using the Quotient Property.	$\dfrac{\sqrt{9p^4q^5}}{\sqrt{16}}$
Simplify the radicals in the numerator and the denominator.	$\dfrac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$
Simplify.	$\dfrac{3p^2q^2\sqrt{q}}{4}$

TRY IT : : 9.55 Simplify: $\sqrt{\dfrac{50x^5y^3}{72x^4y}}$.

TRY IT : : 9.56 Simplify: $\sqrt{\dfrac{48m^7n^2}{125m^5n^9}}$.

9.2 EXERCISES

Practice Makes Perfect

Use the Product Property to Simplify Square Roots

In the following exercises, simplify.

53. $\sqrt{27}$

54. $\sqrt{80}$

55. $\sqrt{125}$

56. $\sqrt{96}$

57. $\sqrt{200}$

58. $\sqrt{147}$

59. $\sqrt{450}$

60. $\sqrt{252}$

61. $\sqrt{800}$

62. $\sqrt{288}$

63. $\sqrt{675}$

64. $\sqrt{1250}$

65. $\sqrt{x^7}$

66. $\sqrt{y^{11}}$

67. $\sqrt{p^3}$

68. $\sqrt{q^5}$

69. $\sqrt{m^{13}}$

70. $\sqrt{n^{21}}$

71. $\sqrt{r^{25}}$

72. $\sqrt{s^{33}}$

73. $\sqrt{49n^{17}}$

74. $\sqrt{25m^9}$

75. $\sqrt{81r^{15}}$

76. $\sqrt{100s^{19}}$

77. $\sqrt{98m^5}$

78. $\sqrt{32n^{11}}$

79. $\sqrt{125r^{13}}$

80. $\sqrt{80s^{15}}$

81. $\sqrt{200p^{13}}$

82. $\sqrt{128q^3}$

83. $\sqrt{242m^{23}}$

84. $\sqrt{175n^{13}}$

85. $\sqrt{147m^7n^{11}}$

86. $\sqrt{48m^7n^5}$

87. $\sqrt{75r^{13}s^9}$

88. $\sqrt{96r^3s^3}$

89. $\sqrt{300p^9q^{11}}$

90. $\sqrt{192q^3r^7}$

91. $\sqrt{242m^{13}n^{21}}$

92. $\sqrt{150m^9n^3}$

93. $5+\sqrt{12}$

94. $8+\sqrt{96}$

95. $1+\sqrt{45}$

96. $3+\sqrt{125}$

97. $\dfrac{10-\sqrt{24}}{2}$

98. $\dfrac{8-\sqrt{80}}{4}$

99. $\dfrac{3+\sqrt{90}}{3}$

100. $\dfrac{15+\sqrt{75}}{5}$

Use the Quotient Property to Simplify Square Roots

In the following exercises, simplify.

101. $\sqrt{\dfrac{49}{64}}$

102. $\sqrt{\dfrac{100}{36}}$

103. $\sqrt{\dfrac{121}{16}}$

104. $\sqrt{\dfrac{144}{169}}$

105. $\sqrt{\dfrac{72}{98}}$

106. $\sqrt{\dfrac{75}{12}}$

107. $\sqrt{\dfrac{45}{125}}$

108. $\sqrt{\dfrac{300}{243}}$

109. $\sqrt{\dfrac{x^{10}}{x^6}}$

110. $\sqrt{\dfrac{p^{20}}{p^{10}}}$

111. $\sqrt{\dfrac{y^4}{y^8}}$

112. $\sqrt{\dfrac{q^8}{q^{14}}}$

113. $\sqrt{\dfrac{200x^7}{2x^3}}$

114. $\sqrt{\dfrac{98y^{11}}{2y^5}}$

115. $\sqrt{\dfrac{96p^9}{6p}}$

116. $\sqrt{\dfrac{108q^{10}}{3q^2}}$

117. $\sqrt{\dfrac{36}{35}}$

118. $\sqrt{\dfrac{144}{65}}$

119. $\sqrt{\dfrac{20}{81}}$

120. $\sqrt{\dfrac{21}{196}}$

121. $\sqrt{\dfrac{96x^7}{121}}$

122. $\sqrt{\dfrac{108y^4}{49}}$

123. $\sqrt{\dfrac{300m^5}{64}}$

124. $\sqrt{\dfrac{125n^7}{169}}$

125. $\sqrt{\dfrac{98r^5}{100}}$

126. $\sqrt{\dfrac{180s^{10}}{144}}$

127. $\sqrt{\dfrac{28q^6}{225}}$

128. $\sqrt{\dfrac{150r^3}{256}}$

129. $\sqrt{\dfrac{75r^9}{s^8}}$

130. $\sqrt{\dfrac{72x^5}{y^6}}$

131. $\sqrt{\dfrac{28p^7}{q^2}}$

132. $\sqrt{\dfrac{45r^3}{s^{10}}}$

133. $\sqrt{\dfrac{100x^5}{36x^3}}$

134. $\sqrt{\dfrac{49r^{12}}{16r^6}}$

135. $\sqrt{\dfrac{121p^5}{81p^2}}$

136. $\sqrt{\dfrac{25r^8}{64r}}$

137. $\sqrt{\dfrac{32x^5y^3}{18x^3y}}$

138. $\sqrt{\dfrac{75r^6s^8}{48rs^4}}$

139. $\sqrt{\dfrac{27p^2q}{108p^5q^3}}$

140. $\sqrt{\dfrac{50r^5s^2}{128r^2s^5}}$

Everyday Math

141.

ⓐ Elliott decides to construct a square garden that will take up 288 square feet of his yard. Simplify $\sqrt{288}$ to determine the length and the width of his garden. Round to the nearest tenth of a foot.

ⓑ Suppose Elliott decides to reduce the size of his square garden so that he can create a 5-foot-wide walking path on the north and east sides of the garden. Simplify $\sqrt{288} - 5$ to determine the length and width of the new garden. Round to the nearest tenth of a foot.

142.

ⓐ Melissa accidentally drops a pair of sunglasses from the top of a roller coaster, 64 feet above the ground. Simplify $\sqrt{\frac{64}{16}}$ to determine the number of seconds it takes for the sunglasses to reach the ground.

ⓑ Suppose the sunglasses in the previous example were dropped from a height of 144 feet. Simplify $\sqrt{\frac{144}{16}}$ to determine the number of seconds it takes for the sunglasses to reach the ground.

Writing Exercises

143. Explain why $\sqrt{x^4} = x^2$. Then explain why $\sqrt{x^{16}} = x^8$.

144. Explain why $7 + \sqrt{9}$ is not equal to $\sqrt{7 + 9}$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
use the Product Property to simplify square roots.			
use the Quotient Property to simplify square roots.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

9.3 Add and Subtract Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract like square roots
- Add and subtract square roots that need simplification

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: ⓐ $3x + 9x$ ⓑ $5m + 5n$.
   If you missed this problem, review **Example 1.24**.
2. Simplify: $\sqrt{50x^3}$.
   If you missed this problem, review **Example 9.16**.

We know that we must follow the order of operations to simplify expressions with square roots. The radical is a grouping symbol, so we work inside the radical first. We simplify $\sqrt{2+7}$ in this way:

	$\sqrt{2+7}$
Add inside the radical.	$\sqrt{9}$
Simplify.	3

So if we have to add $\sqrt{2} + \sqrt{7}$, we must not combine them into one radical.

$$\sqrt{2} + \sqrt{7} \neq \sqrt{2+7}$$

Trying to add square roots with different radicands is like trying to add unlike terms.

But, just like we can add	$x + x,$	we can add	$\sqrt{3} + \sqrt{3}.$
	$x + x = 2x$		$\sqrt{3} + \sqrt{3} = 2\sqrt{3}$

Adding square roots with the same radicand is just like adding like terms. We call square roots with the same radicand like square roots to remind us they work the same as like terms.

Like Square Roots

Square roots with the same radicand are called **like square roots**.

We add and subtract like square roots in the same way we add and subtract like terms. We know that $3x + 8x$ is $11x$. Similarly we add $3\sqrt{x} + 8\sqrt{x}$ and the result is $11\sqrt{x}$.

Add and Subtract Like Square Roots

Think about adding like terms with variables as you do the next few examples. When you have like radicands, you just add or subtract the coefficients. When the radicands are not like, you cannot combine the terms.

EXAMPLE 9.29

Simplify: $2\sqrt{2} - 7\sqrt{2}$.

Solution

	$2\sqrt{2} - 7\sqrt{2}$
Since the radicals are like, we subtract the coefficient	$-5\sqrt{2}$

TRY IT : : 9.57 Simplify: $8\sqrt{2} - 9\sqrt{2}$.

TRY IT : : 9.58 Simplify: $5\sqrt{3} - 9\sqrt{3}$.

EXAMPLE 9.30

Simplify: $3\sqrt{y} + 4\sqrt{y}$.

Solution

	$3\sqrt{y} + 4\sqrt{y}$
Since the radicals are like, we add the coefficient	$7\sqrt{y}$

TRY IT : : 9.59 Simplify: $2\sqrt{x} + 7\sqrt{x}$.

TRY IT : : 9.60 Simplify: $5\sqrt{u} + 3\sqrt{u}$.

EXAMPLE 9.31

Simplify: $4\sqrt{x} - 2\sqrt{y}$.

Solution

	$4\sqrt{x} - 2\sqrt{y}$
Since the radicals are not like, we cannot subtract them. We leave the expression as is.	$4\sqrt{x} - 2\sqrt{y}$

TRY IT : : 9.61 Simplify: $7\sqrt{p} - 6\sqrt{q}$.

TRY IT : : 9.62 Simplify: $6\sqrt{a} - 3\sqrt{b}$.

EXAMPLE 9.32

Simplify: $5\sqrt{13} + 4\sqrt{13} + 2\sqrt{13}$.

Solution

	$5\sqrt{13} + 4\sqrt{13} + 2\sqrt{13}$
Since the radicals are like, we add the coefficient	$11\sqrt{13}$

TRY IT : : 9.63 Simplify: $4\sqrt{11} + 2\sqrt{11} + 3\sqrt{11}$.

TRY IT : : 9.64 Simplify: $6\sqrt{10} + 2\sqrt{10} + 3\sqrt{10}$.

EXAMPLE 9.33

Simplify: $2\sqrt{6} - 6\sqrt{6} + 3\sqrt{3}$.

Solution

	$2\sqrt{6} - 6\sqrt{6} + 3\sqrt{3}$
Since the fir t two radicals are like, we subtract their coefficient	$-4\sqrt{6} + 3\sqrt{3}$

TRY IT : : 9.65 Simplify: $5\sqrt{5} - 4\sqrt{5} + 2\sqrt{6}$.

TRY IT : : 9.66 Simplify: $3\sqrt{7} - 8\sqrt{7} + 2\sqrt{5}$.

EXAMPLE 9.34

Simplify: $2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$.

Solution

	$2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$
Since the radicals are like, we combine them.	$0\sqrt{5n}$
Simplify.	0

TRY IT : : 9.67 Simplify: $\sqrt{7x} - 7\sqrt{7x} + 4\sqrt{7x}$.

TRY IT : : 9.68 Simplify: $4\sqrt{3y} - 7\sqrt{3y} + 2\sqrt{3y}$.

When radicals contain more than one variable, as long as all the variables and their exponents are identical, the radicals are like.

EXAMPLE 9.35

Simplify: $\sqrt{3xy} + 5\sqrt{3xy} - 4\sqrt{3xy}$.

Solution

	$\sqrt{3xy} + 5\sqrt{3xy} - 4\sqrt{3xy}$
Since the radicals are like, we combine them.	$2\sqrt{3xy}$

TRY IT : : 9.69 Simplify: $\sqrt{5xy} + 4\sqrt{5xy} - 7\sqrt{5xy}$.

TRY IT : : 9.70 Simplify: $3\sqrt{7mn} + \sqrt{7mn} - 4\sqrt{7mn}$.

Add and Subtract Square Roots that Need Simplification

Remember that we always simplify square roots by removing the largest perfect-square factor. Sometimes when we have to add or subtract square roots that do not appear to have like radicals, we find like radicals after simplifying the square roots.

EXAMPLE 9.36

Simplify: $\sqrt{20} + 3\sqrt{5}$.

Solution

	$\sqrt{20} + 3\sqrt{5}$
Simplify the radicals, when possible.	$\sqrt{4} \cdot \sqrt{5} + 3\sqrt{5}$ $2\sqrt{5} + 3\sqrt{5}$
Combine the like radicals.	$5\sqrt{5}$

TRY IT : : 9.71 Simplify: $\sqrt{18} + 6\sqrt{2}$.

TRY IT : : 9.72 Simplify: $\sqrt{27} + 4\sqrt{3}$.

EXAMPLE 9.37

Simplify: $\sqrt{48} - \sqrt{75}$.

Solution

	$\sqrt{48} - \sqrt{75}$
Simplify the radicals.	$\sqrt{16} \cdot \sqrt{3} - \sqrt{25} \cdot \sqrt{3}$ $4\sqrt{3} - 5\sqrt{3}$
Combine the like radicals.	$-\sqrt{3}$

TRY IT : : 9.73 Simplify: $\sqrt{32} - \sqrt{18}$.

TRY IT : : 9.74 Simplify: $\sqrt{20} - \sqrt{45}$.

Just like we use the Associative Property of Multiplication to simplify $5(3x)$ and get $15x$, we can simplify $5(3\sqrt{x})$ and get $15\sqrt{x}$. We will use the Associative Property to do this in the next example.

EXAMPLE 9.38

Simplify: $5\sqrt{18} - 2\sqrt{8}$.

Solution

	$5\sqrt{18} - 2\sqrt{8}$
Simplify the radicals.	$5 \cdot \sqrt{9} \cdot \sqrt{2} - 2 \cdot \sqrt{4} \cdot \sqrt{2}$
	$5 \cdot 3 \cdot \sqrt{2} - 2 \cdot 2 \cdot \sqrt{2}$ $15\sqrt{2} - 4\sqrt{2}$
Combine the like radicals.	$11\sqrt{2}$

TRY IT : : 9.75 Simplify: $4\sqrt{27} - 3\sqrt{12}$.

TRY IT : : 9.76 Simplify: $3\sqrt{20} - 7\sqrt{45}$.

EXAMPLE 9.39

Simplify: $\frac{3}{4}\sqrt{192} - \frac{5}{6}\sqrt{108}$.

Solution

	$\frac{3}{4}\sqrt{192} - \frac{5}{6}\sqrt{108}$
Simplify the radicals.	$\frac{3}{4}\sqrt{64}\cdot\sqrt{3} - \frac{5}{6}\sqrt{36}\cdot\sqrt{3}$
	$\frac{3}{4}\cdot 8\cdot\sqrt{3} - \frac{5}{6}\cdot 6\cdot\sqrt{3}$
	$6\sqrt{3} - 5\sqrt{3}$
Combine the like radicals.	$\sqrt{3}$

TRY IT : : 9.77 Simplify: $\frac{2}{3}\sqrt{108} - \frac{5}{7}\sqrt{147}$.

TRY IT : : 9.78 Simplify: $\frac{3}{5}\sqrt{200} - \frac{3}{4}\sqrt{128}$.

EXAMPLE 9.40

Simplify: $\frac{2}{3}\sqrt{48} - \frac{3}{4}\sqrt{12}$.

Solution

	$\frac{2}{3}\sqrt{48} - \frac{3}{4}\sqrt{12}$
Simplify the radicals.	$\frac{2}{3}\sqrt{16}\cdot\sqrt{3} - \frac{3}{4}\sqrt{4}\cdot\sqrt{3}$
	$\frac{2}{3}\cdot 4\cdot\sqrt{3} - \frac{3}{4}\cdot 2\cdot\sqrt{3}$
	$\frac{8}{3}\sqrt{3} - \frac{3}{2}\sqrt{3}$
Find a common denominator to subtract the coefficients of he like radicals.	$\frac{16}{6}\sqrt{3} - \frac{9}{6}\sqrt{3}$
Simplify.	$\frac{7}{6}\sqrt{3}$

TRY IT : : 9.79 Simplify: $\frac{2}{5}\sqrt{32} - \frac{1}{3}\sqrt{8}$.

TRY IT : : 9.80 Simplify: $\frac{1}{3}\sqrt{80} - \frac{1}{4}\sqrt{125}$.

In the next example, we will remove constant and variable factors from the square roots.

EXAMPLE 9.41

Simplify: $\sqrt{18n^5} - \sqrt{32n^5}$.

Solution

	$\sqrt{18n^5} - \sqrt{32n^5}$
Simplify the radicals.	$\sqrt{9n^4} \cdot \sqrt{2n} - \sqrt{16n^4} \cdot \sqrt{2n}$
	$3n^2\sqrt{2n} - 4n^2\sqrt{2n}$
Combine the like radicals.	$-n^2\sqrt{2n}$

TRY IT : : 9.81 Simplify: $\sqrt{32m^7} - \sqrt{50m^7}$.

TRY IT : : 9.82 Simplify: $\sqrt{27p^3} - \sqrt{48p^3}$.

EXAMPLE 9.42

Simplify: $9\sqrt{50m^2} - 6\sqrt{48m^2}$.

Solution

	$9\sqrt{50m^2} - 6\sqrt{48m^2}$
Simplify the radicals.	$9\sqrt{25m^2} \cdot \sqrt{2} - 6\sqrt{16m^2} \cdot \sqrt{3}$
	$9 \cdot 5m \cdot \sqrt{2} - 6 \cdot 4m \cdot \sqrt{3}$
The radicals are not like and so cannot be combined.	$45m\sqrt{2} - 24m\sqrt{3}$

TRY IT : : 9.83 Simplify: $5\sqrt{32x^2} - 3\sqrt{48x^2}$.

TRY IT : : 9.84 Simplify: $7\sqrt{48y^2} - 4\sqrt{72y^2}$.

EXAMPLE 9.43

Simplify: $2\sqrt{8x^2} - 5x\sqrt{32} + 5\sqrt{18x^2}$.

Solution

	$2\sqrt{8x^2} - 5x\sqrt{32} + 5\sqrt{18x^2}$
Simplify the radicals.	$2\sqrt{4x^2} \cdot \sqrt{2} - 5x\sqrt{16} \cdot \sqrt{2} + 5\sqrt{9x^2} \cdot \sqrt{2}$
	$2 \cdot 2x \cdot \sqrt{2} - 5x \cdot 4 \cdot \sqrt{2} + 5 \cdot 3x \cdot \sqrt{2}$
	$4x\sqrt{2} - 20x\sqrt{2} + 15x\sqrt{2}$
Combine the like radicals.	$-x\sqrt{2}$

TRY IT : : 9.85 Simplify: $3\sqrt{12x^2} - 2x\sqrt{48} + 4\sqrt{27x^2}$.

TRY IT : : 9.86 Simplify: $3\sqrt{18x^2} - 6x\sqrt{32} + 2\sqrt{50x^2}$.

MEDIA : :

Access this online resource for additional instruction and practice with the adding and subtracting square roots.

- **Adding/Subtracting Square Roots (https://openstax.org/l/25AddSubtrSR)**

9.3 EXERCISES

Practice Makes Perfect

Add and Subtract Like Square Roots

In the following exercises, simplify.

145. $8\sqrt{2} - 5\sqrt{2}$

146. $7\sqrt{2} - 3\sqrt{2}$

147. $3\sqrt{5} + 6\sqrt{5}$

148. $4\sqrt{5} + 8\sqrt{5}$

149. $9\sqrt{7} - 10\sqrt{7}$

150. $11\sqrt{7} - 12\sqrt{7}$

151. $7\sqrt{y} + 2\sqrt{y}$

152. $9\sqrt{n} + 3\sqrt{n}$

153. $\sqrt{a} - 4\sqrt{a}$

154. $\sqrt{b} - 6\sqrt{b}$

155. $5\sqrt{c} + 2\sqrt{c}$

156. $7\sqrt{d} + 2\sqrt{d}$

157. $8\sqrt{a} - 2\sqrt{b}$

158. $5\sqrt{c} - 3\sqrt{d}$

159. $5\sqrt{m} + \sqrt{n}$

160. $\sqrt{n} + 3\sqrt{p}$

161. $8\sqrt{7} + 2\sqrt{7} + 3\sqrt{7}$

162. $6\sqrt{5} + 3\sqrt{5} + \sqrt{5}$

163. $3\sqrt{11} + 2\sqrt{11} - 8\sqrt{11}$

164. $2\sqrt{15} + 5\sqrt{15} - 9\sqrt{15}$

165. $3\sqrt{3} - 8\sqrt{3} + 7\sqrt{5}$

166. $5\sqrt{7} - 8\sqrt{7} + 6\sqrt{3}$

167. $6\sqrt{2} + 2\sqrt{2} - 3\sqrt{5}$

168. $7\sqrt{5} + \sqrt{5} - 8\sqrt{10}$

169. $3\sqrt{2a} - 4\sqrt{2a} + 5\sqrt{2a}$

170. $\sqrt{11b} - 5\sqrt{11b} + 3\sqrt{11b}$

171. $8\sqrt{3c} + 2\sqrt{3c} - 9\sqrt{3c}$

172. $3\sqrt{5d} + 8\sqrt{5d} - 11\sqrt{5d}$

173. $5\sqrt{3ab} + \sqrt{3ab} - 2\sqrt{3ab}$

174. $8\sqrt{11cd} + 5\sqrt{11cd} - 9\sqrt{11cd}$

175. $2\sqrt{pq} - 5\sqrt{pq} + 4\sqrt{pq}$

176. $11\sqrt{2rs} - 9\sqrt{2rs} + 3\sqrt{2rs}$

Add and Subtract Square Roots that Need Simplification

In the following exercises, simplify.

177. $\sqrt{50} + 4\sqrt{2}$

178. $\sqrt{48} + 2\sqrt{3}$

179. $\sqrt{80} - 3\sqrt{5}$

180. $\sqrt{28} - 4\sqrt{7}$

181. $\sqrt{27} - \sqrt{75}$

182. $\sqrt{72} - \sqrt{98}$

183. $\sqrt{48} + \sqrt{27}$

184. $\sqrt{45} + \sqrt{80}$

185. $2\sqrt{50} - 3\sqrt{72}$

186. $3\sqrt{98} - \sqrt{128}$

187. $2\sqrt{12} + 3\sqrt{48}$

188. $4\sqrt{75} + 2\sqrt{108}$

189. $\frac{2}{3}\sqrt{72} + \frac{1}{5}\sqrt{50}$

190. $\frac{2}{5}\sqrt{75} + \frac{3}{4}\sqrt{48}$

191. $\frac{1}{2}\sqrt{20} - \frac{2}{3}\sqrt{45}$

192. $\frac{2}{3}\sqrt{54} - \frac{3}{4}\sqrt{96}$

193. $\frac{1}{6}\sqrt{27} - \frac{3}{8}\sqrt{48}$

194. $\frac{1}{8}\sqrt{32} - \frac{1}{10}\sqrt{50}$

195. $\frac{1}{4}\sqrt{98} - \frac{1}{3}\sqrt{128}$

196. $\frac{1}{3}\sqrt{24} + \frac{1}{4}\sqrt{54}$

197. $\sqrt{72a^5} - \sqrt{50a^5}$

198. $\sqrt{48b^5} - \sqrt{75b^5}$

199. $\sqrt{80c^7} - \sqrt{20c^7}$

200. $\sqrt{96d^9} - \sqrt{24d^9}$

201. $9\sqrt{80p^4} - 6\sqrt{98p^4}$

202. $8\sqrt{72q^6} - 3\sqrt{75q^6}$

203. $2\sqrt{50r^8} + 4\sqrt{54r^8}$

204. $5\sqrt{27s^6} + 2\sqrt{20s^6}$

205. $3\sqrt{20x^2} - 4\sqrt{45x^2} + 5x\sqrt{80}$

206. $2\sqrt{28x^2} - \sqrt{63x^2} + 6x\sqrt{7}$

207. $3\sqrt{128y^2} + 4y\sqrt{162} - 8\sqrt{98y^2}$

208. $3\sqrt{75y^2} + 8y\sqrt{48} - \sqrt{300y^2}$

Mixed Practice

209. $2\sqrt{8} + 6\sqrt{8} - 5\sqrt{8}$

210. $\frac{2}{3}\sqrt{27} + \frac{3}{4}\sqrt{48}$

211. $\sqrt{175k^4} - \sqrt{63k^4}$

212. $\frac{5}{6}\sqrt{162} + \frac{3}{16}\sqrt{128}$

213. $2\sqrt{363} - 2\sqrt{300}$

214. $\sqrt{150} + 4\sqrt{6}$

215. $9\sqrt{2} - 8\sqrt{2}$

216. $5\sqrt{x} - 8\sqrt{y}$

217. $8\sqrt{13} - 4\sqrt{13} - 3\sqrt{13}$

218. $5\sqrt{12c^4} - 3\sqrt{27c^6}$

219. $\sqrt{80a^5} - \sqrt{45a^5}$

220. $\frac{3}{5}\sqrt{75} - \frac{1}{4}\sqrt{48}$

221. $21\sqrt{19} - 2\sqrt{19}$

222. $\sqrt{500} + \sqrt{405}$

223. $\frac{5}{6}\sqrt{27} + \frac{5}{8}\sqrt{48}$

224. $11\sqrt{11} - 10\sqrt{11}$

225. $\sqrt{75} - \sqrt{108}$

226. $2\sqrt{98} - 4\sqrt{72}$

227. $4\sqrt{24x^2} - \sqrt{54x^2} + 3x\sqrt{6}$

228. $8\sqrt{80y^6} - 6\sqrt{48y^6}$

Everyday Math

229. A decorator decides to use square tiles as an accent strip in the design of a new shower, but she wants to rotate the tiles to look like diamonds. She will use 9 large tiles that measure 8 inches on a side and 8 small tiles that measure 2 inches on a side. $9(8\sqrt{2}) + 8(2\sqrt{2})$. Determine the width of the accent strip by simplifying the expression $9(8\sqrt{2}) + 8(2\sqrt{2})$. (Round to the nearest tenth of an inch.)

230. Suzy wants to use square tiles on the border of a spa she is installing in her backyard. She will use large tiles that have area of 12 square inches, medium tiles that have area of 8 square inches, and small tiles that have area of 4 square inches. Once section of the border will require 4 large tiles, 8 medium tiles, and 10 small tiles to cover the width of the wall. Simplify the expression $4\sqrt{12} + 8\sqrt{8} + 10\sqrt{4}$ to determine the width of the wall.

Writing Exercises

231. Explain the difference between like radicals and unlike radicals. Make sure your answer makes sense for radicals containing both numbers and variables.

232. Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
add and subtract like square roots.			
add and subtract square roots that need simplification.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

9.4 Multiply Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Multiply square roots
- Use polynomial multiplication to multiply square roots

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $(3u)(8v)$.
   If you missed this problem, review Example 6.26.
2. Simplify: $6(12-7n)$.
   If you missed this problem, review Example 6.28.
3. Simplify: $(2+a)(4-a)$.
   If you missed this problem, review Example 6.39.

Multiply Square Roots

We have used the Product Property of Square Roots to simplify square roots by removing the perfect square factors. The Product Property of Square Roots says

$$\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$$

We can use the Product Property of Square Roots 'in reverse' to multiply square roots.

$$\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$$

Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Square Roots so we see both ways together.

Product Property of Square Roots

If *a*, *b* are nonnegative real numbers, then

$$\sqrt{ab}=\sqrt{a}\cdot\sqrt{b} \quad \text{and} \quad \sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$$

So we can multiply $\sqrt{3}\cdot\sqrt{5}$ in this way:

$$\begin{gathered}\sqrt{3}\cdot\sqrt{5}\\ \sqrt{3\cdot 5}\\ \sqrt{15}\end{gathered}$$

Sometimes the product gives us a perfect square:

$$\begin{gathered}\sqrt{2}\cdot\sqrt{8}\\ \sqrt{2\cdot 8}\\ \sqrt{16}\\ 4\end{gathered}$$

Even when the product is not a perfect square, we must look for perfect-square factors and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x\cdot 3y$ we multiply the coefficients together and then the variables. The result is $12xy$. Keep this in mind as you do these examples.

EXAMPLE 9.44

Simplify: ⓐ $\sqrt{2}\cdot\sqrt{6}$ ⓑ $(4\sqrt{3})(2\sqrt{12})$.

Solution

ⓐ

	$\sqrt{2} \cdot \sqrt{6}$
Multiply using the Product Property.	$\sqrt{12}$
Simplify the radical.	$\sqrt{4} \cdot \sqrt{3}$
Simplify.	$2\sqrt{3}$

ⓑ

	$(4\sqrt{3})(2\sqrt{12})$
Multiply using the Product Property.	$8\sqrt{36}$
Simplify the radical.	$8 \cdot 6$
Simplify.	48

Notice that in (b) we multiplied the coefficients and multiplied the radicals. Also, we did not simplify $\sqrt{12}$. We waited to get the product and then simplified.

TRY IT : : 9.87 Simplify: ⓐ $\sqrt{3} \cdot \sqrt{6}$ ⓑ $(2\sqrt{6})(3\sqrt{12})$.

TRY IT : : 9.88 Simplify: ⓐ $\sqrt{5} \cdot \sqrt{10}$ ⓑ $(6\sqrt{3})(5\sqrt{6})$.

EXAMPLE 9.45

Simplify: $(6\sqrt{2})(3\sqrt{10})$.

Solution

	$(6\sqrt{2})(3\sqrt{10})$
Multiply using the Product Property.	$18\sqrt{20}$
Simplify the radical.	$18\sqrt{4} \cdot \sqrt{5}$
Simplify.	$18 \cdot 2 \cdot \sqrt{5}$
	$36\sqrt{5}$

TRY IT : : 9.89 Simplify: $(3\sqrt{2})(2\sqrt{30})$.

TRY IT : : 9.90 Simplify: $(3\sqrt{3})(3\sqrt{6})$.

When we have to multiply square roots, we first find the product and then remove any perfect square factors.

EXAMPLE 9.46

Simplify: ⓐ $\left(\sqrt{8x^3}\right)\left(\sqrt{3x}\right)$ ⓑ $\left(\sqrt{20y^2}\right)\left(\sqrt{5y^3}\right)$.

Solution

ⓐ

	$\left(\sqrt{8x^3}\right)\left(\sqrt{3x}\right)$
Multiply using the Product Property.	$\sqrt{24x^4}$
Simplify the radical.	$\sqrt{4x^4} \cdot \sqrt{6}$
Simplify.	$2x^2\sqrt{6}$

ⓑ

	$\left(\sqrt{20y^2}\right)\left(\sqrt{5y^3}\right)$
Multiply using the Product Property.	$\sqrt{100y^5}$
Simplify the radical.	$10y^2\sqrt{y}$

TRY IT : : 9.91 Simplify: ⓐ $\left(\sqrt{6x^3}\right)\left(\sqrt{3x}\right)$ ⓑ $\left(\sqrt{2y^3}\right)\left(\sqrt{50y^2}\right)$.

TRY IT : : 9.92 Simplify: ⓐ $\left(\sqrt{6x^5}\right)\left(\sqrt{2x}\right)$ ⓑ $\left(\sqrt{12y^2}\right)\left(\sqrt{3y^5}\right)$.

EXAMPLE 9.47

Simplify: $\left(10\sqrt{6p^3}\right)\left(3\sqrt{18p}\right)$.

Solution

	$\left(10\sqrt{6p^3}\right)\left(3\sqrt{18p}\right)$
Multiply.	$30\sqrt{108p^4}$
Simplify the radical.	$30\sqrt{36p^4} \cdot \sqrt{3}$
	$30 \cdot 6p^2 \cdot \sqrt{3}$
	$180p^2\sqrt{3}$

TRY IT : : 9.93 Simplify: $\left(6\sqrt{2x^2}\right)\left(8\sqrt{45x^4}\right)$.

TRY IT : : 9.94 Simplify: $\left(2\sqrt{6y^4}\right)\left(12\sqrt{30y}\right)$.

EXAMPLE 9.48

Simplify: ⓐ $(\sqrt{2})^2$ ⓑ $(-\sqrt{11})^2$.

Solution

ⓐ

	$(\sqrt{2})^2$
Rewrite as a product.	$(\sqrt{2})(\sqrt{2})$
Multiply.	$\sqrt{4}$
Simplify.	2

ⓑ

	$(-\sqrt{11})^2$
Rewrite as a product.	$(-\sqrt{11})(-\sqrt{11})$
Multiply.	$\sqrt{121}$
Simplify.	11

TRY IT : : 9.95 Simplify: ⓐ $(\sqrt{12})^2$ ⓑ $(-\sqrt{15})^2$.

TRY IT : : 9.96 Simplify: ⓐ $(\sqrt{16})^2$ ⓑ $(-\sqrt{20})^2$.

The results of the previous example lead us to this property.

Squaring a Square Root

If *a* is a nonnegative real number, then

$$(\sqrt{a})^2 = a$$

By realizing that squaring and taking a square root are 'opposite' operations, we can simplify $(\sqrt{2})^2$ and get 2 right away. When we multiply the two like square roots in part (a) of the next example, it is the same as squaring.

EXAMPLE 9.49

Simplify: ⓐ $(2\sqrt{3})(8\sqrt{3})$ ⓑ $(3\sqrt{6})^2$.

Solution

ⓐ

	$(2\sqrt{3})(8\sqrt{3})$
Multiply. Remember, $(\sqrt{3})^2 = 3$.	$16 \cdot 3$
Simplify.	48

ⓑ

	$\left(3\sqrt{6}\right)^2$
Multiply.	$9 \cdot 6$
Simplify.	54

> TRY IT : : 9.97 Simplify: ⓐ $\left(6\sqrt{11}\right)\left(5\sqrt{11}\right)$ ⓑ $\left(5\sqrt{8}\right)^2$.

> TRY IT : : 9.98 Simplify: ⓐ $(3\sqrt{7})(10\sqrt{7})$ ⓑ $\left(-4\sqrt{6}\right)^2$.

Use Polynomial Multiplication to Multiply Square Roots

In the next few examples, we will use the Distributive Property to multiply expressions with square roots.
We will first distribute and then simplify the square roots when possible.

EXAMPLE 9.50

Simplify: ⓐ $3\left(5-\sqrt{2}\right)$ ⓑ $\sqrt{2}\left(4-\sqrt{10}\right)$.

Solution

ⓐ

	$3\left(5-\sqrt{2}\right)$
Distribute.	$15-3\sqrt{2}$

ⓑ

	$\sqrt{2}\left(4-\sqrt{10}\right)$
Distribute.	$4\sqrt{2}-\sqrt{20}$
	$4\sqrt{2}-2\sqrt{5}$

> TRY IT : : 9.99 Simplify: ⓐ $2\left(3-\sqrt{5}\right)$ ⓑ $\sqrt{3}\left(2-\sqrt{18}\right)$.

> TRY IT : : 9.100 Simplify: ⓐ $6\left(2+\sqrt{6}\right)$ ⓑ $\sqrt{7}\left(1+\sqrt{14}\right)$.

EXAMPLE 9.51

Simplify: ⓐ $\sqrt{5}\left(7+2\sqrt{5}\right)$ ⓑ $\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$.

Solution

ⓐ

	$\sqrt{5}(7+2\sqrt{5})$
Multiply.	$7\sqrt{5}+2\cdot 5$
Simplify.	$7\sqrt{5}+10$
	$10+7\sqrt{5}$

ⓑ

	$\sqrt{6}(\sqrt{2}+\sqrt{18})$
Multiply.	$\sqrt{12}+\sqrt{108}$
Simplify.	$\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$ $2\sqrt{3}+6\sqrt{3}$
Combine like radicals.	$8\sqrt{3}$

TRY IT : : 9.101 Simplify: ⓐ $\sqrt{6}(1+3\sqrt{6})$ ⓑ $\sqrt{12}(\sqrt{3}+\sqrt{24})$.

TRY IT : : 9.102 Simplify: ⓐ $\sqrt{8}(2-5\sqrt{8})$ ⓑ $\sqrt{14}(\sqrt{2}+\sqrt{42})$.

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

EXAMPLE 9.52

Simplify: $(2+\sqrt{3})(4-\sqrt{3})$.

Solution

	$(2+\sqrt{3})(4-\sqrt{3})$
Multiply.	$8-2\sqrt{3}+4\sqrt{3}-3$
Combine like terms.	$5+2\sqrt{3}$

TRY IT : : 9.103 Simplify: $(1+\sqrt{6})(3-\sqrt{6})$.

TRY IT : : 9.104 Simplify: $(4-\sqrt{10})(2+\sqrt{10})$.

EXAMPLE 9.53

Simplify: $(3-2\sqrt{7})(4-2\sqrt{7})$.

Solution

	$(3-2\sqrt{7})(4-2\sqrt{7})$
Multiply.	$12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$
Simplify.	$12-6\sqrt{7}-8\sqrt{7}+28$
Combine like terms.	$40-14\sqrt{7}$

TRY IT : : 9.105 Simplify: $(6-3\sqrt{7})(3+4\sqrt{7})$.

TRY IT : : 9.106 Simplify: $(2-3\sqrt{11})(4-\sqrt{11})$.

EXAMPLE 9.54

Simplify: $(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$.

Solution

	$(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$
Multiply.	$3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$
Simplify.	$6+12\sqrt{10}-\sqrt{10}-20$
Combine like terms.	$-14+11\sqrt{10}$

TRY IT : : 9.107 Simplify: $(5\sqrt{3}-\sqrt{7})(\sqrt{3}+2\sqrt{7})$.

TRY IT : : 9.108 Simplify: $(\sqrt{6}-3\sqrt{8})(2\sqrt{6}+\sqrt{8})$

EXAMPLE 9.55

Simplify: $(4-2\sqrt{x})(1+3\sqrt{x})$.

Solution

	$(4-2\sqrt{x})(1+3\sqrt{x})$
Multiply.	$4+12\sqrt{x}-2\sqrt{x}-6x$
Combine like terms.	$4+10\sqrt{x}-6x$

TRY IT : : 9.109 Simplify: $(6-5\sqrt{m})(2+3\sqrt{m})$.

TRY IT : : 9.110 Simplify: $(10+3\sqrt{n})(1-5\sqrt{n})$.

Note that some special products made our work easier when we multiplied binomials earlier. This is true when we multiply square roots, too. The special product formulas we used are shown below.

Special Product Formulas

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)(a+b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	

We will use the special product formulas in the next few examples. We will start with the Binomial Squares formula.

EXAMPLE 9.56

Simplify: ⓐ $(2+\sqrt{3})^2$ ⓑ $(4-2\sqrt{5})^2$.

Solution

Be sure to include the $2ab$ term when squaring a binomial.

ⓐ

	$\overset{(a\ +\ b)^2}{(2+\sqrt{3})^2}$
Multiply using the binomial square pattern.	$\underbrace{2^2}_{a^2} + \underbrace{2\cdot 2\cdot\sqrt{3}}_{2ab} + \underbrace{(\sqrt{3})^2}_{b^2}$
Simplify.	$4+4\sqrt{3}+3$
Combine like terms.	$7+4\sqrt{3}$

ⓑ

	$\overset{(a\ -\ b)^2}{(4-2\sqrt{5})^2}$
Multiply using the binomial square pattern.	$\underbrace{4^2}_{a^2} - \underbrace{2\cdot 4\cdot 2\sqrt{5}}_{2ab} + \underbrace{(2\sqrt{5})^2}_{b^2}$
Simplify.	$16-16\sqrt{5}+4\cdot 5$ $16-16\sqrt{5}+20$
Combine like terms.	$36-16\sqrt{5}$

TRY IT : : 9.111 Simplify: ⓐ $(10+\sqrt{2})^2$ ⓑ $(1+3\sqrt{6})^2$.

TRY IT : : 9.112 Simplify: ⓐ $(6-\sqrt{5})^2$ ⓑ $(9-2\sqrt{10})^2$.

EXAMPLE 9.57

Simplify: $(1+3\sqrt{x})^2$.

Solution

	$\overset{(a\ +\ \ b)^2}{\left(1+3\sqrt{x}\right)^2}$
Multiply using the binomial square pattern.	$\overset{a^2\ +\quad 2ab\quad +\quad b^2}{1^2+2\cdot 1\cdot 3\sqrt{x}+\left(3\sqrt{x}\right)^2}$
Simplify.	$1+6\sqrt{x}+9x$

> **TRY IT : : 9.113** Simplify: $\left(2+5\sqrt{m}\right)^2$.

> **TRY IT : : 9.114** Simplify: $\left(3-4\sqrt{n}\right)^2$.

In the next two examples, we will find the product of conjugates.

EXAMPLE 9.58

Simplify: $\left(4-\sqrt{2}\right)\left(4+\sqrt{2}\right)$.

Solution

	$\overset{(a\ -\ \ b)\ (a\ +\ \ b)}{\left(4-\sqrt{2}\right)\left(4+\sqrt{2}\right)}$
Multiply using the binomial square pattern.	$\overset{a^2\ -\ \ b^2}{4^2-\left(\sqrt{2}\right)^2}$
Simplify.	$16-2$ 14

> **TRY IT : : 9.115** Simplify: $\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)$.

> **TRY IT : : 9.116** Simplify: $\left(1+\sqrt{5}\right)\left(1-\sqrt{5}\right)$.

EXAMPLE 9.59

Simplify: $\left(5-2\sqrt{3}\right)\left(5+2\sqrt{3}\right)$.

Solution

	$\overset{(a\ -\ \ b)\ (a\ +\ \ b)}{\left(5-2\sqrt{3}\right)\left(5+2\sqrt{3}\right)}$
Multiply using the binomial square pattern.	$\overset{a^2\ -\ \ b^2}{5^2-\left(2\sqrt{3}\right)^2}$
Simplify.	$25-4\cdot 3$ 13

> **TRY IT : : 9.117** Simplify: $\left(3-2\sqrt{5}\right)\left(3+2\sqrt{5}\right)$.

TRY IT : : 9.118 Simplify: $\left(4+5\sqrt{7}\right)\left(4-5\sqrt{7}\right)$.

MEDIA : :

Access these online resources for additional instruction and practice with multiplying square roots.

- **Product Property (https://openstax.org/l/25ProductProp)**
- **Multiply Binomials with Square Roots (https://openstax.org/l/25MultPolySR)**

9.4 EXERCISES

Practice Makes Perfect

Multiply Square Roots

In the following exercises, simplify.

233.
ⓐ $\sqrt{2} \cdot \sqrt{8}$
ⓑ $(3\sqrt{3})(2\sqrt{18})$

234.
ⓐ $\sqrt{6} \cdot \sqrt{6}$
ⓑ $(3\sqrt{2})(2\sqrt{32})$

235.
ⓐ $\sqrt{7} \cdot \sqrt{14}$
ⓑ $(4\sqrt{8})(5\sqrt{8})$

236.
ⓐ $\sqrt{6} \cdot \sqrt{12}$
ⓑ $(2\sqrt{5})(2\sqrt{10})$

237. $(5\sqrt{2})(3\sqrt{6})$

238. $(2\sqrt{3})(4\sqrt{6})$

239. $(-2\sqrt{3})(3\sqrt{18})$

240. $(-4\sqrt{5})(5\sqrt{10})$

241. $(5\sqrt{6})(-\sqrt{12})$

242. $(6\sqrt{2})(-\sqrt{10})$

243. $(-2\sqrt{7})(-2\sqrt{14})$

244. $(-2\sqrt{11})(-4\sqrt{22})$

245.
ⓐ $\left(\sqrt{15y}\right)\left(\sqrt{5y^3}\right)$
ⓑ $\left(\sqrt{2n^2}\right)\left(\sqrt{18n^3}\right)$

246.
ⓐ $\left(\sqrt{14x^3}\right)\left(\sqrt{7x^3}\right)$
ⓑ $\left(\sqrt{3q^2}\right)\left(\sqrt{48q^3}\right)$

247.
ⓐ $\left(\sqrt{16y^2}\right)\left(\sqrt{8y^4}\right)$
ⓑ $\left(\sqrt{11s^6}\right)\left(\sqrt{11s}\right)$

248.
ⓐ $\left(\sqrt{8x^3}\right)\left(\sqrt{3x}\right)$
ⓑ $\left(\sqrt{7r}\right)\left(\sqrt{7r^8}\right)$

249. $\left(2\sqrt{5b^3}\right)\left(4\sqrt{15b}\right)$

250. $\left(3\sqrt{8c^5}\right)\left(2\sqrt{6c^3}\right)$

251. $\left(6\sqrt{3d^3}\right)\left(4\sqrt{12d^5}\right)$

252. $\left(2\sqrt{5b^3}\right)\left(4\sqrt{15b}\right)$

253. $\left(6\sqrt{3d^3}\right)\left(4\sqrt{12d^5}\right)$

254. $\left(-2\sqrt{7z^3}\right)\left(3\sqrt{14z^8}\right)$

255. $\left(4\sqrt{2k^5}\right)\left(-3\sqrt{32k^6}\right)$

256.
ⓐ $(\sqrt{7})^2$
ⓑ $(-\sqrt{15})^2$

257.
ⓐ $(\sqrt{11})^2$
ⓑ $(-\sqrt{21})^2$

258.
ⓐ $(\sqrt{19})^2$
ⓑ $(-\sqrt{5})^2$

259.
ⓐ $(\sqrt{23})^2$
ⓑ $(-\sqrt{3})^2$

260.
ⓐ $(4\sqrt{11})(-3\sqrt{11})$
ⓑ $(5\sqrt{3})^2$

261.
ⓐ $(2\sqrt{13})(-9\sqrt{13})$
ⓑ $(6\sqrt{5})^2$

262.
ⓐ $(-3\sqrt{12})(-2\sqrt{6})$
ⓑ $(-4\sqrt{10})^2$

263.
ⓐ $(-7\sqrt{5})(-3\sqrt{10})$
ⓑ $(-2\sqrt{14})^2$

Use Polynomial Multiplication to Multiply Square Roots

In the following exercises, simplify.

264.
ⓐ $3\left(4-\sqrt{3}\right)$
ⓑ $\sqrt{2}\left(4-\sqrt{6}\right)$

265.
ⓐ $4\left(6-\sqrt{11}\right)$
ⓑ $\sqrt{2}\left(5-\sqrt{12}\right)$

266.
ⓐ $5(3-\sqrt{7})$
ⓑ $\sqrt{3}\left(4-\sqrt{15}\right)$

267.
ⓐ $7(-2-\sqrt{11})$
ⓑ $\sqrt{7}\left(6-\sqrt{14}\right)$

268.
ⓐ $\sqrt{7}\left(5+2\sqrt{7}\right)$
ⓑ $\sqrt{5}\left(\sqrt{10}+\sqrt{18}\right)$

269.
ⓐ $\sqrt{11}\left(8+4\sqrt{11}\right)$
ⓑ $\sqrt{3}\left(\sqrt{12}+\sqrt{27}\right)$

270.
ⓐ $\sqrt{11}\left(-3+4\sqrt{11}\right)$
ⓑ $\sqrt{3}\left(\sqrt{15}-\sqrt{18}\right)$

271.
ⓐ $\sqrt{2}\left(-5+9\sqrt{2}\right)$
ⓑ $\sqrt{7}\left(\sqrt{3}-\sqrt{21}\right)$

272. $\left(8+\sqrt{3}\right)\left(2-\sqrt{3}\right)$

273. $\left(7+\sqrt{3}\right)\left(9-\sqrt{3}\right)$

274. $\left(8-\sqrt{2}\right)\left(3+\sqrt{2}\right)$

275. $\left(9-\sqrt{2}\right)\left(6+\sqrt{2}\right)$

276. $(3-\sqrt{7})(5-\sqrt{7})$

277. $(5-\sqrt{7})(4-\sqrt{7})$

278. $\left(1+3\sqrt{10}\right)\left(5-2\sqrt{10}\right)$

279. $\left(7-2\sqrt{5}\right)\left(4+9\sqrt{5}\right)$

280. $\left(\sqrt{3}+\sqrt{10}\right)\left(\sqrt{3}+2\sqrt{10}\right)$

281. $\left(\sqrt{11}+\sqrt{5}\right)\left(\sqrt{11}+6\sqrt{5}\right)$

282. $\left(2\sqrt{7}-5\sqrt{11}\right)\left(4\sqrt{7}+9\sqrt{11}\right)$

283. $\left(4\sqrt{6}+7\sqrt{13}\right)\left(8\sqrt{6}-3\sqrt{13}\right)$

284. $(5-\sqrt{u})(3+\sqrt{u})$

285. $(9-\sqrt{w})(2+\sqrt{w})$

286. $(7+2\sqrt{m})(4+9\sqrt{m})$

287. $(6+5\sqrt{n})(11+3\sqrt{n})$

288.
ⓐ $\left(3+\sqrt{5}\right)^2$
ⓑ $\left(2-5\sqrt{3}\right)^2$

289.
ⓐ $\left(4+\sqrt{11}\right)^2$
ⓑ $\left(3-2\sqrt{5}\right)^2$

290.
ⓐ $\left(9-\sqrt{6}\right)^2$
ⓑ $\left(10+3\sqrt{7}\right)^2$

291.
ⓐ $\left(5-\sqrt{10}\right)^2$
ⓑ $\left(8+3\sqrt{2}\right)^2$

292. $\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)$

293. $\left(10-\sqrt{3}\right)\left(10+\sqrt{3}\right)$

294. $\left(4+\sqrt{2}\right)\left(4-\sqrt{2}\right)$

295. $\left(7+\sqrt{10}\right)\left(7-\sqrt{10}\right)$

296. $\left(4+9\sqrt{3}\right)\left(4-9\sqrt{3}\right)$

297. $\left(1+8\sqrt{2}\right)\left(1-8\sqrt{2}\right)$

298. $\left(12-5\sqrt{5}\right)\left(12+5\sqrt{5}\right)$

299. $\left(9-4\sqrt{3}\right)\left(9+4\sqrt{3}\right)$

Mixed Practice

In the following exercises, simplify.

300. $\sqrt{3}\cdot\sqrt{21}$

301. $\left(4\sqrt{6}\right)\left(-\sqrt{18}\right)$

302. $\left(-5+\sqrt{7}\right)\left(6+\sqrt{21}\right)$

303. $(-5\sqrt{7})\left(6\sqrt{21}\right)$

304. $(-4\sqrt{2})\left(2\sqrt{18}\right)$

305. $\left(\sqrt{35y^3}\right)\left(\sqrt{7y^3}\right)$

306. $\left(4\sqrt{12x^5}\right)\left(2\sqrt{6x^3}\right)$

307. $\left(\sqrt{29}\right)^2$

308. $(-4\sqrt{17})\left(-3\sqrt{17}\right)$

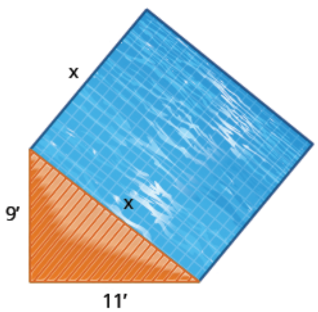
309. $\left(-4+\sqrt{17}\right)\left(-3+\sqrt{17}\right)$

Everyday Math

310. A landscaper wants to put a square reflecting pool next to a triangular deck, as shown below. The triangular deck is a right triangle, with legs of length 9 feet and 11 feet, and the pool will be adjacent to the hypotenuse.

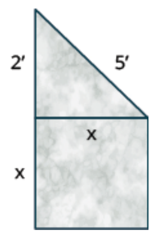
ⓐ Use the Pythagorean Theorem to find the length of a side of the pool. Round your answer to the nearest tenth of a foot.

ⓑ Find the exact area of the pool.



311. An artist wants to make a small monument in the shape of a square base topped by a right triangle, as shown below. The square base will be adjacent to one leg of the triangle. The other leg of the triangle will measure 2 feet and the hypotenuse will be 5 feet.

ⓐ Use the Pythagorean Theorem to find the length of a side of the square base. Round your answer to the nearest tenth of a foot.



ⓑ Find the exact area of the face of the square base.

312. A square garden will be made with a stone border on one edge. If only $3+\sqrt{10}$ feet of stone are available, simplify $\left(3+\sqrt{10}\right)^2$ to determine the area of the largest such garden.

313. A garden will be made so as to contain two square sections, one section with side length $\sqrt{5}+\sqrt{6}$ yards and one section with side length $\sqrt{2}+\sqrt{3}$ yards. Simplify $\left(\sqrt{5}+\sqrt{6}\right)\left(\sqrt{2}+\sqrt{3}\right)$ to determine the total area of the garden.

314. Suppose a third section will be added to the garden in the previous exercise. The third section is to have a width of $\sqrt{432}$ feet. Write an expression that gives the total area of the garden.

Writing Exercises

315.

ⓐ Explain why $(-\sqrt{n})^2$ is always positive, for $n \geq 0$.

ⓑ Explain why $(-\sqrt{n})^2$ is always negative, for $n \geq 0$.

316. Use the binomial square pattern to simplify $\left(3+\sqrt{2}\right)^2$. Explain all your steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
multiply square roots.			
use polynomial multiplication to multiply square roots.			

ⓑ *On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you* *improve this?*

9.5 Divide Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Divide square roots
- Rationalize a one-term denominator
- Rationalize a two-term denominator

Be Prepared!

Before you get started, take this readiness quiz.

1. Find a fraction equivalent to $\frac{5}{8}$ with denominator 48.

   If you missed this problem, review Example 1.64.
2. Simplify: $\left(\sqrt{5}\right)^2$.

   If you missed this problem, review Example 9.48.
3. Multiply: $(7+3x)(7-3x)$.

   If you missed this problem, review Example 6.54.

Divide Square Roots

We know that we simplify fractions by removing factors common to the numerator and the denominator. When we have a fraction with a square root in the numerator, we first simplify the square root. Then we can look for common factors.

Common Factors	No common factors
$\frac{3\sqrt{2}}{3\cdot 5}$	$\frac{2\sqrt{3}}{3\cdot 5}$

EXAMPLE 9.60

Simplify: $\frac{\sqrt{54}}{6}$.

Solution

	$\frac{\sqrt{54}}{6}$
Simplify the radical.	$\frac{\sqrt{9}\cdot\sqrt{6}}{6}$
Simplify.	$\frac{3\sqrt{6}}{6}$
Remove the common factors.	$\frac{\cancel{3}\sqrt{6}}{\cancel{3}\cdot 2}$
Simplify.	$\frac{\sqrt{6}}{2}$

TRY IT : : 9.119 Simplify: $\frac{\sqrt{32}}{8}$.

TRY IT : : 9.120 Simplify: $\frac{\sqrt{75}}{15}$.

EXAMPLE 9.61

Simplify: $\dfrac{6-\sqrt{24}}{12}$.

Solution

	$\dfrac{6-\sqrt{24}}{12}$
Simplify the radical.	$\dfrac{6-\sqrt{4}\cdot\sqrt{6}}{12}$
Simplify.	$\dfrac{6-2\sqrt{6}}{12}$
Factor the common factor from the numerator.	$\dfrac{2(3-\sqrt{6})}{2\cdot 6}$
Remove the common factors.	$\dfrac{\cancel{2}(3-\sqrt{6})}{\cancel{2}\cdot 6}$
Simplify.	$\dfrac{3-\sqrt{6}}{6}$

TRY IT : : 9.121 Simplify: $\dfrac{8-\sqrt{40}}{10}$.

TRY IT : : 9.122 Simplify: $\dfrac{10-\sqrt{75}}{20}$.

We have used the Quotient Property of Square Roots to simplify square roots of fractions. The Quotient Property of Square Roots says

$$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}},\ b\neq 0$$

Sometimes we will need to use the Quotient Property of Square Roots 'in reverse' to simplify a fraction with square roots.

$$\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}},\ b\neq 0$$

We will rewrite the Quotient Property of Square Roots so we see both ways together. Remember: we assume all variables are greater than or equal to zero so that their square roots are real numbers.

Quotient Property of Square Roots

If *a*, *b* are non-negative real numbers and $b\neq 0$, then

$$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}\quad\text{and}\quad\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$$

We will use the Quotient Property of Square Roots 'in reverse' when the fraction we start with is the quotient of two square roots, and neither radicand is a perfect square. When we write the fraction in a single square root, we may find common factors in the numerator and denominator.

EXAMPLE 9.62

Simplify: $\dfrac{\sqrt{27}}{\sqrt{75}}$.

Solution

	$\dfrac{\sqrt{27}}{\sqrt{75}}$
Neither radicand is a perfect square, so rewrite using the quotient property of square roots.	$\sqrt{\dfrac{27}{75}}$
Remove common factors in the numerator and denominator.	$\sqrt{\dfrac{\cancel{3} \cdot 9}{\cancel{3} \cdot 25}}$
Simplify.	$\sqrt{\dfrac{9}{25}}$
	$\dfrac{3}{5}$

TRY IT : : 9.123 Simplify: $\dfrac{\sqrt{48}}{\sqrt{108}}$.

TRY IT : : 9.124 Simplify: $\dfrac{\sqrt{96}}{\sqrt{54}}$.

We will use the Quotient Property for Exponents, $\dfrac{a^m}{a^n} = a^{m-n}$, when we have variables with exponents in the radicands.

EXAMPLE 9.63

Simplify: $\dfrac{\sqrt{6y^5}}{\sqrt{2y}}$.

Solution

	$\dfrac{\sqrt{6y^5}}{\sqrt{2y}}$
Neither radicand is a perfect square, so rewrite using the quotient property of square roots.	$\sqrt{\dfrac{6y^5}{2y}}$
Remove common factors in the numerator and denominator.	$\sqrt{\dfrac{\cancel{2} \cdot 3 \cdot y^4 \cdot \cancel{y}}{\cancel{2} \cdot \cancel{y}}}$
Simplify.	$\sqrt{3y^4}$
Simplify the radical.	$y^2\sqrt{3}$

TRY IT : : 9.125 Simplify: $\dfrac{\sqrt{12r^3}}{\sqrt{6r}}$.

TRY IT : : 9.126

Simplify: $\dfrac{\sqrt{14p^9}}{\sqrt{2p^5}}$.

EXAMPLE 9.64

Simplify: $\dfrac{\sqrt{72x^3}}{\sqrt{162x}}$.

Solution

	$\dfrac{\sqrt{72x^3}}{\sqrt{162x}}$
Rewrite using the quotient property of square roots.	$\sqrt{\dfrac{72x^3}{162x}}$
Remove common factors.	$\sqrt{\dfrac{\cancel{18}\cdot 4\cdot x^2\cdot \cancel{x}}{\cancel{18}\cdot 9\cdot \cancel{x}}}$
Simplify.	$\sqrt{\dfrac{4x^2}{9}}$
Simplify the radical.	$\dfrac{2x}{3}$

TRY IT : : 9.127

Simplify: $\dfrac{\sqrt{50s^3}}{\sqrt{128s}}$.

TRY IT : : 9.128

Simplify: $\dfrac{\sqrt{75q^5}}{\sqrt{108q}}$.

EXAMPLE 9.65

Simplify: $\dfrac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$.

Solution

	$\dfrac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$
Rewrite using the quotient property of square roots.	$\sqrt{\dfrac{147ab^8}{3a^3b^4}}$
Remove common factors.	$\sqrt{\dfrac{49b^4}{a^2}}$
Simplify the radical.	$\dfrac{7b^2}{a}$

TRY IT : : 9.129 Simplify: $\dfrac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$.

TRY IT : : 9.130 Simplify: $\dfrac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$.

Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, tables of square roots were used to find approximate values of square roots. Figure 9.3 shows a portion of a table of squares and square roots. Square roots are approximated to five decimal places in this table.

n	n^2	$\sqrt{n}$
200	40,000	14.14214
201	40,401	14.17745
202	40,804	14.21267
203	41,209	14.24781
204	41,616	14.28286
205	42,025	14.31782
206	42,436	14.35270
207	42,849	14.38749
208	43,264	14.42221
209	43,681	14.45683
210	44,100	14.49138

Figure 9.3 A table of square roots was used to find approximate values of square roots before there were calculators.

If someone needed to approximate a fraction with a square root in the denominator, it meant doing long division with a five decimal-place divisor. This was a very cumbersome process.

For this reason, a process called rationalizing the denominator was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. This process is still used today and is useful in other areas of mathematics, too.

Rationalizing the Denominator

The process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer is called **rationalizing the denominator**.

Square roots of numbers that are not perfect squares are irrational numbers. When we **rationalize the denominator**, we write an equivalent fraction with a rational number in the denominator.

Let's look at a numerical example.

Suppose we need an approximate value for the fraction.	$\dfrac{1}{\sqrt{2}}$
A fi e decimal place approximation to $\sqrt{2}$ is 1.41421.	$\dfrac{1}{1.41421}$
Without a calculator, would you want to do this division?	$1.41421\overline{)1.0}$

But we can find a fraction equivalent to $\dfrac{1}{\sqrt{2}}$ by multiplying the numerator and denominator by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}}$$

$$\frac{1\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$$

$$\frac{\sqrt{2}}{2}$$

Now if we need an approximate value, we divide $2\overline{)1.41421}$. This is much easier.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a square root.

Similarly, a square root is not considered simplified if the radicand contains a fraction.

Simplified Square Roots

A square root is considered simplified if there are

- no perfect-square factors in the radicand
- no fractions in the radicand
- no square roots in the denominator of a fraction

To rationalize a denominator, we use the property that $(\sqrt{a})^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

EXAMPLE 9.66

Simplify: $\frac{4}{\sqrt{3}}$.

Solution

To rationalize a denominator, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

	$\frac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$.	$\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$
Simplify.	$\frac{4\sqrt{3}}{3}$

TRY IT : : 9.131 Simplify: $\frac{5}{\sqrt{3}}$.

TRY IT : : 9.132 Simplify: $\frac{6}{\sqrt{5}}$.

EXAMPLE 9.67

Simplify: $-\frac{8}{3\sqrt{6}}$.

Solution

To remove the square root from the denominator, we multiply it by itself. To keep the fractions equivalent, we multiply both the numerator and denominator by $\sqrt{6}$.

	$-\frac{8}{3\sqrt{6}}$
Multiply both the numerator and the denominator by $\sqrt{6}$.	$-\frac{8 \cdot \sqrt{6}}{3\sqrt{6} \cdot \sqrt{6}}$

Simplify.	$-\frac{8\sqrt{6}}{3\cdot 6}$
Remove common factors.	$-\frac{4\cdot\cancel{2}\sqrt{6}}{3\cdot\cancel{2}\cdot 3}$
Simplify.	$-\frac{4\sqrt{6}}{9}$

TRY IT : : 9.133 Simplify: $\frac{5}{2\sqrt{5}}$.

TRY IT : : 9.134 Simplify: $-\frac{9}{4\sqrt{3}}$.

Always simplify the radical in the denominator first, before you rationalize it. This way the numbers stay smaller and easier to work with.

EXAMPLE 9.68

Simplify: $\sqrt{\frac{5}{12}}$.

Solution

	$\sqrt{\frac{5}{12}}$
The fraction is not a perfect square, so rewrite using the Quotient Property.	$\frac{\sqrt{5}}{\sqrt{12}}$
Simplify the denominator	$\frac{\sqrt{5}}{2\sqrt{3}}$
Rationalize the denominator.	$\frac{\sqrt{5}\cdot\sqrt{3}}{2\sqrt{3}\cdot\sqrt{3}}$
Simplify.	$\frac{\sqrt{15}}{2\cdot 3}$
Simplify.	$\frac{\sqrt{15}}{6}$

TRY IT : : 9.135 Simplify: $\sqrt{\frac{7}{18}}$.

TRY IT : : 9.136 Simplify: $\sqrt{\frac{3}{32}}$.

EXAMPLE 9.69

Simplify: $\sqrt{\frac{11}{28}}$.

Solution

	$\sqrt{\frac{11}{28}}$
Rewrite using the Quotient Property.	$\frac{\sqrt{11}}{\sqrt{28}}$
Simplify the denominator.	$\frac{\sqrt{11}}{2\sqrt{7}}$
Rationalize the denominator.	$\frac{\sqrt{11} \cdot \sqrt{7}}{2\sqrt{7} \cdot \sqrt{7}}$
Simplify.	$\frac{\sqrt{77}}{2 \cdot 7}$
Simplify.	$\frac{\sqrt{77}}{14}$

TRY IT : : 9.137 Simplify: $\sqrt{\frac{3}{27}}$.

TRY IT : : 9.138 Simplify: $\sqrt{\frac{10}{50}}$.

Rationalize a Two-Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates pattern to rationalize the denominator.

$$(a - b)(a + b) \qquad (2 - \sqrt{5})(2 + \sqrt{5})$$
$$a^2 - b^2 \qquad 2^2 - (\sqrt{5})^2$$
$$4 - 5$$
$$-1$$

When we multiply a binomial that includes a square root by its conjugate, the product has no square roots.

EXAMPLE 9.70

Simplify: $\frac{4}{4 + \sqrt{2}}$.

Solution

	$\frac{4}{4+\sqrt{2}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{4(4-\sqrt{2})}{(4+\sqrt{2})(4-\sqrt{2})}$
Multiply the conjugates in the denominator.	$\frac{4(4-\sqrt{2})}{4^2-(\sqrt{2})^2}$
Simplify the denominator.	$\frac{4(4-\sqrt{2})}{16-2}$

Simplify the denominator.	$\frac{4(4-\sqrt{2})}{14}$
Remove common factors from the numerator and denominator.	$\frac{2(4-\sqrt{2})}{7}$
We leave the numerator in factored form to make it easier to look for common factors after we have simplified the denominator.	

TRY IT : : 9.139 Simplify: $\frac{2}{2+\sqrt{3}}$.

TRY IT : : 9.140 Simplify: $\frac{5}{5+\sqrt{3}}$.

EXAMPLE 9.71

Simplify: $\frac{5}{2-\sqrt{3}}$.

Solution

	$\frac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$
Multiply the conjugates in the denominator.	$\frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{4-3}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{1}$
Simplify.	$5(2+\sqrt{3})$

TRY IT : : 9.141 Simplify: $\frac{3}{1-\sqrt{5}}$.

TRY IT : : 9.142 Simplify: $\frac{2}{4-\sqrt{6}}$.

EXAMPLE 9.72

Simplify: $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution

	$\dfrac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\dfrac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}-\sqrt{6}\right)\left(\sqrt{u}+\sqrt{6}\right)}$
Multiply the conjugates in the denominator.	$\dfrac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}\right)^2-\left(\sqrt{6}\right)^2}$
Simplify the denominator.	$\dfrac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{u-6}$

TRY IT : : 9.143 Simplify: $\dfrac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

TRY IT : : 9.144 Simplify: $\dfrac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$.

EXAMPLE 9.73

Simplify: $\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Solution

	$\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\dfrac{\left(\sqrt{x}+\sqrt{7}\right)\left(\sqrt{x}+\sqrt{7}\right)}{\left(\sqrt{x}-\sqrt{7}\right)\left(\sqrt{x}+\sqrt{7}\right)}$
Multiply the conjugates in the denominator.	$\dfrac{\left(\sqrt{x}+\sqrt{7}\right)\left(\sqrt{x}+\sqrt{7}\right)}{\left(\sqrt{x}\right)^2-\left(\sqrt{7}\right)^2}$
Simplify the denominator.	$\dfrac{\left(\sqrt{x}+\sqrt{7}\right)^2}{x-7}$
We do not square the numerator. In factored form, we can see there are no common factors to remove from the numerator and denominator.	

TRY IT : : 9.145 Simplify: $\dfrac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

TRY IT : : 9.146 Simplify: $\dfrac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$.

MEDIA : :

Access this online resource for additional instruction and practice with dividing and rationalizing.

- **Dividing and Rationalizing (https://openstax.org/l/25DivideRation)**

9.5 EXERCISES

Practice Makes Perfect

Divide Square Roots

In the following exercises, simplify.

317. $\frac{\sqrt{27}}{6}$

318. $\frac{\sqrt{50}}{10}$

319. $\frac{\sqrt{72}}{9}$

320. $\frac{\sqrt{243}}{6}$

321. $\frac{2-\sqrt{32}}{8}$

322. $\frac{3+\sqrt{27}}{9}$

323. $\frac{6+\sqrt{45}}{6}$

324. $\frac{10-\sqrt{200}}{20}$

325. $\frac{\sqrt{80}}{\sqrt{125}}$

326. $\frac{\sqrt{72}}{\sqrt{200}}$

327. $\frac{\sqrt{128}}{\sqrt{72}}$

328. $\frac{\sqrt{48}}{\sqrt{75}}$

329. ⓐ $\frac{\sqrt{8x^6}}{\sqrt{2x^2}}$ ⓑ $\frac{\sqrt{200m^5}}{\sqrt{98m}}$

330. ⓐ $\frac{\sqrt{10y^3}}{\sqrt{5y}}$ ⓑ $\frac{\sqrt{108n^7}}{\sqrt{243n^3}}$

331. $\frac{\sqrt{75r^3}}{\sqrt{108r}}$

332. $\frac{\sqrt{196q^5}}{\sqrt{484q}}$

333. $\frac{\sqrt{108p^5q^2}}{\sqrt{34p^3q^6}}$

334. $\frac{\sqrt{98rs^{10}}}{\sqrt{2r^3s^4}}$

335. $\frac{\sqrt{320mn^5}}{\sqrt{45m^7n^3}}$

336. $\frac{\sqrt{810c^3d^7}}{\sqrt{1000c^5d}}$

337. $\frac{\sqrt{98}}{14}$

338. $\frac{\sqrt{72}}{18}$

339. $\frac{5+\sqrt{125}}{15}$

340. $\frac{6-\sqrt{45}}{12}$

341. $\frac{\sqrt{96}}{\sqrt{150}}$

342. $\frac{\sqrt{28}}{\sqrt{63}}$

343. $\frac{\sqrt{26y^7}}{\sqrt{2y}}$

344. $\frac{\sqrt{15x^3}}{\sqrt{3x}}$

Rationalize a One-Term Denominator

In the following exercises, simplify and rationalize the denominator.

345. $\frac{10}{\sqrt{6}}$

346. $\frac{8}{\sqrt{3}}$

347. $\frac{6}{\sqrt{7}}$

348. $\frac{4}{\sqrt{5}}$

349. $\frac{3}{\sqrt{13}}$

350. $\frac{10}{\sqrt{11}}$

351. $\frac{10}{3\sqrt{10}}$

352. $\frac{2}{5\sqrt{2}}$

353. $\frac{4}{9\sqrt{5}}$

354. $\frac{9}{2\sqrt{7}}$

355. $-\frac{9}{2\sqrt{3}}$

356. $-\frac{8}{3\sqrt{6}}$

357. $\sqrt{\frac{3}{20}}$

358. $\sqrt{\frac{4}{27}}$

359. $\sqrt{\frac{7}{40}}$

360. $\sqrt{\frac{8}{45}}$

361. $\sqrt{\frac{19}{175}}$

362. $\sqrt{\frac{17}{192}}$

Rationalize a Two-Term Denominator

In the following exercises, simplify by rationalizing the denominator.

363. ⓐ $\frac{3}{3+\sqrt{11}}$ ⓑ $\frac{8}{1-\sqrt{5}}$

364. ⓐ $\frac{4}{4+\sqrt{7}}$ ⓑ $\frac{7}{2-\sqrt{6}}$

365. ⓐ $\frac{5}{5+\sqrt{6}}$ ⓑ $\frac{6}{3-\sqrt{7}}$

366. ⓐ $\frac{6}{6+\sqrt{5}}$ ⓑ $\frac{5}{4-\sqrt{11}}$

367. $\frac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

368. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

369. $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{6}}$

370. $\frac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

371. $\frac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

372. $\frac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

373. $\frac{\sqrt{150x^2y^6}}{\sqrt{6x^4y^2}}$

374. $\frac{\sqrt{80p^3q}}{\sqrt{5pq^5}}$

375. $\frac{15}{\sqrt{5}}$

376. $\frac{3}{5\sqrt{8}}$

377. $\sqrt{\frac{8}{54}}$

378. $\sqrt{\frac{12}{20}}$

379. $\frac{3}{5+\sqrt{5}}$

380. $\frac{20}{4-\sqrt{3}}$

381. $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{3}}$

382. $\frac{\sqrt{5}}{\sqrt{y}-\sqrt{7}}$

383. $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

384. $\frac{\sqrt{m}-\sqrt{3}}{\sqrt{m}+\sqrt{3}}$

Everyday Math

385. A supply kit is dropped from an airplane flying at an altitude of 250 feet. Simplify $\sqrt{\frac{250}{16}}$ to determine how many seconds it takes for the supply kit to reach the ground.

386. A flare is dropped into the ocean from an airplane flying at an altitude of 1,200 feet. Simplify $\sqrt{\frac{1200}{16}}$ to determine how many seconds it takes for the flare to reach the ocean.

Writing Exercises

387.

ⓐ Simplify $\sqrt{\frac{27}{3}}$ and explain all your steps.

ⓑ Simplify $\sqrt{\frac{27}{5}}$ and explain all your steps.

ⓒ Why are the two methods of simplifying square roots different?

388.

ⓐ Approximate $\frac{1}{\sqrt{2}}$ by dividing $\frac{1}{1.414}$ using long division without a calculator.

ⓑ Rationalizing the denominator of $\frac{1}{\sqrt{2}}$ gives $\frac{\sqrt{2}}{2}$. Approximate $\frac{\sqrt{2}}{2}$ by dividing $\frac{1.414}{2}$ using long division without a calculator.

ⓒ Do you agree that rationalizing the denominator makes calculations easier? Why or why not?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
divide square roots.			
rationalize a one-term denominator.			
rationalize a two-term denominator.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

9.6 Solve Equations with Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Use square roots in applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: ⓐ $\sqrt{9}$ ⓑ 9^2.
   If you missed this problem, review Example 9.1 and Example 1.19.
2. Solve: $5(x+1)-4=3(2x-7)$.
   If you missed this problem, review Example 2.42.
3. Solve: $n^2-6n+8=0$.
   If you missed this problem, review Example 7.73.

Solve Radical Equations

In this section we will solve equations that have the variable in the radicand of a square root. Equations of this type are called radical equations.

Radical Equation

An equation in which the variable is in the radicand of a square root is called a **radical equation**.

As usual, in solving these equations, what we do to one side of an equation we must do to the other side as well. Since squaring a quantity and taking a square root are 'opposite' operations, we will square both sides in order to remove the radical sign and solve for the variable inside.

But remember that when we write $\sqrt{a}$ we mean the principal square root. So $\sqrt{a} \geq 0$ always. When we solve radical equations by squaring both sides we may get an algebraic solution that would make $\sqrt{a}$ negative. This algebraic solution would not be a solution to the original radical equation; it is an *extraneous solution.* We saw extraneous solutions when we solved rational equations, too.

EXAMPLE 9.74

For the equation $\sqrt{x+2}=x$:

ⓐ Is $x=2$ a solution? ⓑ Is $x=-1$ a solution?

Solution

ⓐ Is $x=2$ a solution?

	$\sqrt{x+2}=x$
Let $x = 2$.	$\sqrt{2+2}\stackrel{?}{=}2$
Simplify.	$\sqrt{4}\stackrel{?}{=}2$
	$2=2\checkmark$
	2 is a solution.

ⓑ Is $x=-1$ a solution?

	$\sqrt{x+2} = x$
Let $x = -1$.	$\sqrt{-1+2} \stackrel{?}{=} -1$
Simplify.	$\sqrt{1} \stackrel{?}{=} -1$
	$1 \neq -1$
	-1 is not a solution.
	-1 is an extraneous solution to the equation.

TRY IT : : 9.147 For the equation $\sqrt{x+6} = x$:

ⓐ Is $x = -2$ a solution? ⓑ Is $x = 3$ a solution?

TRY IT : : 9.148 For the equation $\sqrt{-x+2} = x$:

ⓐ Is $x = -2$ a solution? ⓑ Is $x = 1$ a solution?

Now we will see how to solve a radical equation. Our strategy is based on the relation between taking a square root and squaring.

$$\text{For } a \geq 0, \quad (\sqrt{a})^2 = a$$

EXAMPLE 9.75 HOW TO SOLVE RADICAL EQUATIONS

Solve: $\sqrt{2x-1} = 7$.

Solution

Step 1. Isolate the radical on one side of the equation.	$\sqrt{2x-1}$ is already isolated on the left side.	$\sqrt{2x-1} = 7$
Step 2. Square both sides of the equation.	Remember, $\left(\sqrt{a}\right)^2 = a$	$\left(\sqrt{2x-1}\right)^2 = (7)^2$
Step 3. Solve the new equation.		$2x - 1 = 49$ $2x = 50$ $x = 25$
Step 4. Check the answer.		Check: $\sqrt{2x-1} = 7$ $\sqrt{2(25)-1} \stackrel{?}{=} 7$ $\sqrt{50-1} \stackrel{?}{=} 7$ $\sqrt{49} \stackrel{?}{=} 7$ $7 = 7$ ✓ The solution is $x = 25$.

TRY IT : : 9.149 Solve: $\sqrt{3x-5} = 5$.

TRY IT : : 9.150 Solve: $\sqrt{4x+8} = 6$.

HOW TO : : SOLVE A RADICAL EQUATION.

Step 1. Isolate the radical on one side of the equation.
Step 2. Square both sides of the equation.
Step 3. Solve the new equation.
Step 4. Check the answer.

EXAMPLE 9.76

Solve: $\sqrt{5n-4}-9=0$.

Solution

	$\sqrt{5n-4}-9=0$
To isolate the radical, add 9 to both sides.	$\sqrt{5n-4}-9+9=0+9$
Simplify.	$\sqrt{5n-4}=9$
Square both sides of the equation.	$\left(\sqrt{5n-4}\right)^2=(9)^2$
Solve the new equation.	$5n-4=81$
	$5n=85$
	$n=17$
Check the answer.	
$\sqrt{5n-4}-9=0$ $\sqrt{5(17)-4}-9\stackrel{?}{=}0$ $\sqrt{85-4}-9\stackrel{?}{=}0$ $\sqrt{81}-9\stackrel{?}{=}0$ $9-9\stackrel{?}{=}0$ $0=0\checkmark$	
	The solution is $n=17$.

TRY IT : : 9.151 Solve: $\sqrt{3m+2}-5=0$.

TRY IT : : 9.152 Solve: $\sqrt{10z+1}-2=0$.

EXAMPLE 9.77

Solve: $\sqrt{3y+5}+2=5$.

Solution

	$\sqrt{3y+5}+2=5$
To isolate the radical, subtract 2 from both sides.	$\sqrt{3y+5}+2-2=5-2$
Simplify.	$\sqrt{3y+5}=3$
Square both sides of the equation.	$\left(\sqrt{3y+5}\right)^2=(3)^2$
Solve the new equation.	$3y+5=9$
	$3y=4$
	$y=\frac{4}{3}$
Check the answer.	

$$\begin{aligned}\sqrt{3y+5}+2&=5\\ \sqrt{3\left(\frac{4}{3}\right)+5}+2&\stackrel{?}{=}5\\ \sqrt{4+5}+2&\stackrel{?}{=}5\\ \sqrt{9}+2&\stackrel{?}{=}5\\ 3+2&\stackrel{?}{=}5\\ 5&=5\checkmark\end{aligned}$$

The solution is $y=\frac{4}{3}$.

TRY IT : : 9.153 Solve: $\sqrt{3p+3}+3=5$.

TRY IT : : 9.154 Solve: $\sqrt{5q+1}+4=6$.

When we use a radical sign, we mean the principal or positive root. If an equation has a square root equal to a negative number, that equation will have no solution.

EXAMPLE 9.78

Solve: $\sqrt{9k-2}+1=0$.

Solution

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 from both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$
Since the square root is equal to a negative number, the equation has no solution.	

TRY IT : : 9.155 Solve: $\sqrt{2r-3}+5=0$.

TRY IT : : 9.156 Solve: $\sqrt{7s-3}+2=0$.

If one side of the equation is a binomial, we use the binomial squares formula when we square it.

Binomial Squares

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (a-b)^2 = a^2 - 2ab + b^2$$

Don't forget the middle term!

EXAMPLE 9.79

Solve: $\sqrt{p-1}+1=p$.

Solution

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$
Use the zero product property.	$0=p-1 \quad 0=p-2$
Solve each equation.	$p=1 \quad p=2$
Check the answers.	

$p=1$
$$\sqrt{p-1}+1=p$$
$$\sqrt{1-1}+1\stackrel{?}{=}1$$
$$\sqrt{0}+1\stackrel{?}{=}1$$
$$1=1\checkmark$$

$p=2$
$$\sqrt{p-1}+1=p$$
$$\sqrt{2-1}+1\stackrel{?}{=}2$$
$$\sqrt{1}+1\stackrel{?}{=}2$$
$$2=2\checkmark$$

The solutions are $p=1$, $p=2$.

TRY IT : : 9.157 Solve: $\sqrt{x-2}+2=x$.

TRY IT : : 9.158 Solve: $\sqrt{y-5}+5=y$.

EXAMPLE 9.80

Solve: $\sqrt{r+4}-r+2=0$.

Solution

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$\left(\sqrt{r+4}\right)^2=(r-2)^2$
Solve the new equation.	$r+4=r^2-4r+4$
It is a quadratic equation, so get zero on one side.	$0=r^2-5r$
Factor the right side.	$0=r(r-5)$
Use the zero product property.	$0=r \quad 0=r-5$
Solve the equation.	$r=0 \quad r=5$
Check the answer.	
$r=0$: $\sqrt{r+4}-r+2=0$; $\sqrt{0+4}-0+2\stackrel{?}{=}0$; $\sqrt{4}+2\stackrel{?}{=}0$; $4\neq 0$ — $r=5$: $\sqrt{r+4}-r+2=0$; $\sqrt{5+4}-5+2\stackrel{?}{=}0$; $\sqrt{9}-3\stackrel{?}{=}0$; $0=0\checkmark$	The solution is $r=5$.
	$r=0$ is an extraneous solution.

TRY IT : : 9.159 Solve: $\sqrt{m+9}-m+3=0$.

TRY IT : : 9.160 Solve: $\sqrt{n+1}-n+1=0$.

When there is a coefficient in front of the radical, we must square it, too.

EXAMPLE 9.81

Solve: $3\sqrt{3x-5}-8=4$.

Solution

	$3\sqrt{3x-5}-8=4$
Isolate the radical.	$3\sqrt{3x-5}=12$
Square both sides of the equation.	$\left(3\sqrt{3x-5}\right)^2=(12)^2$
Simplify, then solve the new equation.	$9(3x-5)=144$
Distribute.	$27x-45=144$
Solve the equation.	$27x=189$
	$x=7$
Check the answer.	
$x=7$ $\quad 3\sqrt{3x-5}-8=4$ $3\sqrt{3(7)-5}-8\stackrel{?}{=}4$ $3\sqrt{21-5}-8\stackrel{?}{=}4$ $3\sqrt{16}-8\stackrel{?}{=}4$ $3(4)-8\stackrel{?}{=}4$ $4=4\checkmark$	The solution is $x=7$.

TRY IT : : 9.161 Solve: $2\sqrt{4a+2}-16=16$.

TRY IT : : 9.162 Solve: $3\sqrt{6b+3}-25=50$.

EXAMPLE 9.82

Solve: $\sqrt{4z-3}=\sqrt{3z+2}$.

Solution

	$\sqrt{4z-3}=\sqrt{3z+2}$
The radical terms are isolated.	$\sqrt{4z-3}=\sqrt{3z+2}$
Square both sides of the equation.	$\left(\sqrt{4z-3}\right)^2=\left(\sqrt{3z+2}\right)^2$
Simplify, then solve the new equation.	$4z-3=3z+2$ $z-3=2$ $z=5$
Check the answer. We leave it to you to show that 5 checks!	The solution is $z=5$.

TRY IT : : 9.163 Solve: $\sqrt{2x-5}=\sqrt{5x+3}$.

TRY IT : : 9.164 Solve: $\sqrt{7y+1} = \sqrt{2y-5}$.

Sometimes after squaring both sides of an equation, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and square both sides of the equation again.

EXAMPLE 9.83

Solve: $\sqrt{m}+1 = \sqrt{m+9}$.

Solution

	$\sqrt{m}+1 = \sqrt{m+9}$
The radical on the right side is isolated. Square both sides.	$(\sqrt{m}+1)^2 = \left(\sqrt{m+9}\right)^2$
Simplify—be very careful as you multiply!	$m+2\sqrt{m}+1 = m+9$
There is still a radical in the equation.	
So we must repeat the previous steps. Isolate the radical.	$2\sqrt{m} = 8$
Square both sides.	$(2\sqrt{m})^2 = (8)^2$
Simplify, then solve the new equation.	$4m = 64$
	$m = 16$
Check the answer.	
We leave it to you to show that $m = 16$ checks!	The solution is $m = 16$.

TRY IT : : 9.165 Solve: $\sqrt{x}+3 = \sqrt{x+5}$.

TRY IT : : 9.166 Solve: $\sqrt{m}+5 = \sqrt{m+16}$.

EXAMPLE 9.84

Solve: $\sqrt{q-2}+3 = \sqrt{4q+1}$.

Solution

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right side is isolated. Square both sides.	$\left(\sqrt{q-2}+3\right)^2=\left(\sqrt{4q+1}\right)^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides.	$\left(6\sqrt{q-2}\right)^2=(3q-6)^2$
Simplify, then solve the new equation.	$36(q-2)=9q^2-36q+36$
Distribute.	$36q-72=9q^2-36q+36$
It is a quadratic equation, so get zero on one side.	$0=9q^2-72q+108$
Factor the right side.	$0=9\left(q^2-8q+12\right)$ $0=9(q-6)(q-2)$
Use the zero product property.	$q-6=0 \quad q-2=0$ $q=6 \quad q=2$
The checks are left to you. (Both solutions should work.)	The solutions are $q=6$ and $q=2$.

TRY IT : : 9.167 Solve: $\sqrt{y-3}+2=\sqrt{4y+2}$.

TRY IT : : 9.168 Solve: $\sqrt{n-4}+5=\sqrt{3n+3}$.

Use Square Roots in Applications

As you progress through your college courses, you'll encounter formulas that include square roots in many disciplines. We have already used formulas to solve geometry applications.

We will use our Problem Solving Strategy for Geometry Applications, with slight modifications, to give us a plan for solving applications with formulas from any discipline.

HOW TO : : SOLVE APPLICATIONS WITH FORMULAS.

Step 1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for by choosing a variable to represent it.

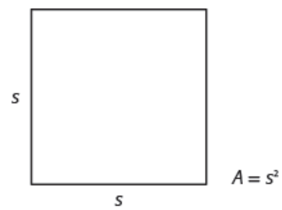
Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

Step 5. **Solve the equation** using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

We used the formula $A = L \cdot W$ to find the area of a rectangle with length *L* and width *W*. A square is a rectangle in which the length and width are equal. If we let *s* be the length of a side of a square, the area of the square is s^2 .

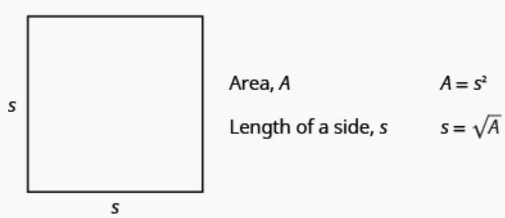


The formula $A = s^2$ gives us the area of a square if we know the length of a side. What if we want to find the length of a side for a given area? Then we need to solve the equation for *s*.

	$A = s^2$
Take the square root of both sides.	$\sqrt{A} = \sqrt{s^2}$
Simplify.	$\sqrt{A} = s$

We can use the formula $s = \sqrt{A}$ to find the length of a side of a square for a given area.

Area of a Square

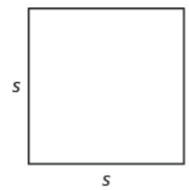


We will show an example of this in the next example.

EXAMPLE 9.85

Mike and Lychelle want to make a square patio. They have enough concrete to pave an area of 200 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of the patio. Round your answer to the nearest tenth of a foot.

Solution

Step 1. Read the problem. Draw a figure and label it with the given information.	
	A = 200 square feet
Step 2. Identify what you are looking for.	The length of a side of the square patio.
Step 3. Name what you are looking for by choosing a variable to represent it.	Let s = the length of a side.
Step 4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute the given information.	$s = \sqrt{A}$, and $A = 200$ $s = \sqrt{200}$
Step 5. Solve the equation using good algebra techniques. Round to one decimal place.	$s = 14.14213...$ $s \approx 14.1$
Step 6. Check the answer in the problem and make sure it makes sense.	
$14.1^2 \stackrel{?}{\approx} 200$ $14.1^2 \approx 198.81$ ✓	
This is close enough because we rounded the square root. Is a patio with side 14.1 feet reasonable? Yes.	
Step 7. Answer the question with a complete sentence.	Each side of the patio should be 14.1 feet.

TRY IT : : 9.169

Katie wants to plant a square lawn in her front yard. She has enough sod to cover an area of 370 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of her lawn. Round your answer to the nearest tenth of a foot.

TRY IT : : 9.170

Sergio wants to make a square mosaic as an inlay for a table he is building. He has enough tile to cover an area of 2704 square centimeters. Use the formula $s = \sqrt{A}$ to find the length of each side of his mosaic. Round your answer to the nearest tenth of a foot.

Another application of square roots has to do with gravity.

Falling Objects

On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula,

$$t = \frac{\sqrt{h}}{4}$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h = 64$ into the formula.

	$t = \frac{\sqrt{h}}{4}$
	$t = \frac{\sqrt{64}}{4}$
Take the square root of 64.	$t = \frac{8}{4}$
Simplify the fraction.	$t = 2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

EXAMPLE 9.86

Christy dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	The time it takes for the sunglasses to reach the river.
Step 3. Name what you are looking for by choosing a variable to represent it.	Let t = time.
Step 4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.	$t = \frac{\sqrt{h}}{4}$, and $h = 400$ $t = \frac{\sqrt{400}}{4}$
Step 5. Solve the equation using good algebra techniques.	$t = \frac{20}{4}$ $t = 5$
Step 6. Check the answer in the problem and make sure it makes sense.	
$5 \stackrel{?}{=} \frac{\sqrt{400}}{4}$ $5 \stackrel{?}{=} \frac{20}{4}$ $5 = 5\ \checkmark$	
Does 5 seconds seem reasonable? Yes.	
Step 7. Answer the question with a complete sentence.	It will take 5 seconds for the sunglasses to hit the water.

TRY IT : : 9.171

A helicopter dropped a rescue package from a height of 1,296 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

TRY IT : : 9.172

A window washer dropped a squeegee from a platform 196 feet above the sidewalk Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Skid Marks and Speed of a Car

If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula,

$$s = \sqrt{24d}$$

EXAMPLE 9.87

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution

Step 1. **Read** the problem.	
Step 2. **Identify** what we are looking for.	The speed of a car.
Step 3. **Name** what we are looking for.	Let s = the speed.
Step 4. **Translate** into an equation by writing the appropriate formula.	$s = \sqrt{24d}$, and $d = 190$
Substitute the given information.	$s = \sqrt{24(190)}$
Step 5. **Solve the equation**.	$s = \sqrt{4560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
Step 6. **Check** the answer in the problem. $67.5 \overset{?}{\approx} \sqrt{24(190)}$ $67.5 \overset{?}{\approx} \sqrt{4560}$ $67.5 \overset{?}{\approx} 67.5277...$	
Is 67.5 mph a reasonable speed?	Yes.
Step 7. **Answer** the question with a complete sentence.	The speed of the car was approximately 67.5 miles per hour.

TRY IT : : 9.173

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

TRY IT : : 9.174

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

9.6 EXERCISES

Practice Makes Perfect

Solve Radical Equations

In the following exercises, check whether the given values are solutions.

389. For the equation $\sqrt{x+12}=x$:

ⓐ Is $x=4$ a solution?

ⓑ Is $x=-3$ a solution?

390. For the equation $\sqrt{-y+20}=y$:

ⓐ Is $y=4$ a solution?

ⓑ Is $y=-5$ a solution?

391. For the equation $\sqrt{t+6}=t$:

ⓐ Is $t=-2$ a solution?

ⓑ Is $t=3$ a solution?

392. For the equation $\sqrt{u+42}=u$:

ⓐ Is $u=-6$ a solution?

ⓑ Is $u=7$ a solution?

In the following exercises, solve.

393. $\sqrt{5y+1}=4$

394. $\sqrt{7z+15}=6$

395. $\sqrt{5x-6}=8$

396. $\sqrt{4x-3}=7$

397. $\sqrt{2m-3}-5=0$

398. $\sqrt{2n-1}-3=0$

399. $\sqrt{6v-2}-10=0$

400. $\sqrt{4u+2}-6=0$

401. $\sqrt{5q+3}-4=0$

402. $\sqrt{4m+2}+2=6$

403. $\sqrt{6n+1}+4=8$

404. $\sqrt{2u-3}+2=0$

405. $\sqrt{5v-2}+5=0$

406. $\sqrt{3z-5}+2=0$

407. $\sqrt{2m+1}+4=0$

408.
ⓐ $\sqrt{u-3}+3=u$
ⓑ $\sqrt{x+1}-x+1=0$

409.
ⓐ $\sqrt{v-10}+10=v$
ⓑ $\sqrt{y+4}-y+2=0$

410.
ⓐ $\sqrt{r-1}-r=-1$
ⓑ $\sqrt{z+100}-z+10=0$

411.
ⓐ $\sqrt{s-8}-s=-8$
ⓑ $\sqrt{w+25}-w+5=0$

412. $3\sqrt{2x-3}-20=7$

413. $2\sqrt{5x+1}-8=0$

414. $2\sqrt{8r+1}-8=2$

415. $3\sqrt{7y+1}-10=8$

416. $\sqrt{3u-2}=\sqrt{5u+1}$

417. $\sqrt{4v+3}=\sqrt{v-6}$

418. $\sqrt{8+2r}=\sqrt{3r+10}$

419. $\sqrt{12c+6}=\sqrt{10-4c}$

420.
ⓐ $\sqrt{a}+2=\sqrt{a+4}$
ⓑ $\sqrt{b-2}+1=\sqrt{3b+2}$

421.
ⓐ $\sqrt{r}+6=\sqrt{r+8}$
ⓑ $\sqrt{s-3}+2=\sqrt{s+4}$

422.
ⓐ $\sqrt{u}+1=\sqrt{u+4}$
ⓑ $\sqrt{n-5}+4=\sqrt{3n+7}$

423.
ⓐ $\sqrt{x}+10=\sqrt{x+2}$
ⓑ $\sqrt{y-2}+2=\sqrt{2y+4}$

424. $\sqrt{2y+4}+6=0$

425. $\sqrt{8u+1}+9=0$

426. $\sqrt{a}+1=\sqrt{a+5}$

427. $\sqrt{d}-2=\sqrt{d-20}$

428. $\sqrt{6s+4}=\sqrt{8s-28}$

429. $\sqrt{9p+9}=\sqrt{10p-6}$

Use Square Roots in Applications

In the following exercises, solve. Round approximations to one decimal place.

430. Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

431. Landscaping Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.

432. Gravity While putting up holiday decorations, Renee dropped a light bulb from the top of a 64 foot tall tree. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the light bulb to reach the ground.

433. Gravity An airplane dropped a flare from a height of 1024 feet above a lake. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the flare to reach the water.

434. Gravity A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.

435. Gravity A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.

436. Accident investigation The skid marks for a car involved in an accident measured 54 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

437. Accident investigation The skid marks for a car involved in an accident measured 216 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

438. Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

439. Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 117 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Writing Exercises

440. Explain why an equation of the form $\sqrt{x}+1=0$ has no solution.

441.

ⓐ Solve the equation $\sqrt{r+4}-r+2=0$.

ⓑ Explain why one of the “solutions” that was found was not actually a solution to the equation.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
use square roots in applications.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

9.7 Higher Roots

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with higher roots
- Use the Product Property to simplify expressions with higher roots
- Use the Quotient Property to simplify expressions with higher roots
- Add and subtract higher roots

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $y^5 y^4$.
   If you missed this problem, review **Example 6.18**.
2. Simplify: $\left(n^2\right)^6$.
   If you missed this problem, review **Example 6.22**.
3. Simplify: $\frac{x^8}{x^3}$.
   If you missed this problem, review **Example 6.59**.

Simplify Expressions with Higher Roots

Up to now, in this chapter we have worked with squares and square roots. We will now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.

We write:	We say:
n^2	n squared
n^3	n cubed
n^4	n to the fourth
n^5	n to the fi th

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from -5 to 5. See **Figure 9.4**.

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
–1	1	–1	1	–1
–2	4	–8	16	–32
–3	9	–27	81	–243
–4	16	–64	256	–1024
–5	25	–125	625	–3125

Figure 9.4 First through fifth powers of integers from -5 to 5.

Notice the signs in **Figure 9.4**. All powers of positive numbers are positive, of course. But when we have a negative number, the even powers are positive and the odd powers are negative. We'll copy the row with the powers of -2 below to help you see this.

n	n^2	n^3	n^4	n^5
–2	4	–8	16	–32

Earlier in this chapter we defined the square root of a number.

$$\text{If } n^2 = m, \quad \text{then } n \text{ is a square root of } m.$$

And we have used the notation $\sqrt{m}$ to denote the **principal square root**. So $\sqrt{m} \geq 0$ always.

We will now extend the definition to higher roots.

*n*th Root of a Number

If $b^n = a$, then b is an ***n*th root of a number** a.

The principal *n*th root of a is written $\sqrt[n]{a}$.

n is called the **index** of the radical.

We do not write the index for a square root. Just like we use the word 'cubed' for b^3, we use the term 'cube root' for $\sqrt[3]{a}$.

We refer to **Figure 9.4** to help us find higher roots.

$$\begin{aligned} 4^3 &= 64 & \qquad \sqrt[3]{64} &= 4 \\ 3^4 &= 81 & \sqrt[4]{81} &= 3 \\ (-2)^5 &= -32 & \sqrt[5]{-32} &= -2 \end{aligned}$$

Could we have an even root of a negative number? No. We know that the square root of a negative number is not a real number. The same is true for any even root. Even roots of negative numbers are not real numbers. Odd roots of negative numbers are real numbers.

Properties of $\sqrt[n]{a}$

When n is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number
- $a < 0$, then $\sqrt[n]{a}$ is not a real number

When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of a.

EXAMPLE 9.88

Simplify: ⓐ $\sqrt[3]{8}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{32}$.

Solution

ⓐ

	$\sqrt[3]{8}$
Since $(2)^3 = 8$.	2

ⓑ

	$\sqrt[4]{81}$
Since $(3)^4 = 81$.	3

ⓒ

	$\sqrt[5]{32}$
Since $(2)^5 = 32$.	2

TRY IT : : 9.175 Simplify: ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{256}$ ⓒ $\sqrt[5]{243}$.

TRY IT : : 9.176 Simplify: ⓐ $\sqrt[3]{1000}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{32}$.

EXAMPLE 9.89

Simplify: ⓐ $\sqrt[3]{-64}$ ⓑ $\sqrt[4]{-16}$ ⓒ $\sqrt[5]{-243}$.

Solution

ⓐ

	$\sqrt[3]{-64}$
Since $(-4)^3 = -64$.	-4

ⓑ

	$\sqrt[4]{-16}$
Think, $(\,?\,)^4 = -16$. No real number raised to the fourth power is positive.	Not a real number.

ⓒ

	$\sqrt[5]{-243}$
Since $(-3)^5 = -243$.	-3

TRY IT : : 9.177 Simplify: ⓐ $\sqrt[3]{-125}$ ⓑ $\sqrt[4]{-16}$ ⓒ $\sqrt[5]{-32}$.

TRY IT : : 9.178 Simplify: ⓐ $\sqrt[3]{-216}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1024}$.

When we worked with square roots that had variables in the radicand, we restricted the variables to non-negative values. Now we will remove this restriction.

The odd root of a number can be either positive or negative. We have seen that $\sqrt[3]{-64} = -4$.

But the even root of a non-negative number is always non-negative, because we take the principal *n*th root.

Suppose we start with $a = -5$.

$$(-5)^4 = 625 \qquad \sqrt[4]{625} = 5$$

How can we make sure the fourth root of −5 raised to the fourth power, $(-5)^4$ is 5? We will see in the following property.

Simplifying Odd and Even Roots

For any integer $n \geq 2$,

$$\text{when } n \text{ is odd} \quad \sqrt[n]{a^n} = a$$
$$\text{when } n \text{ is even} \quad \sqrt[n]{a^n} = |a|$$

We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

EXAMPLE 9.90

Simplify: ⓐ $\sqrt{x^2}$ ⓑ $\sqrt[3]{n^3}$ ⓒ $\sqrt[4]{p^4}$ ⓓ $\sqrt[5]{y^5}$.

Solution

We use the absolute value to be sure to get the positive root.

ⓐ

	$\sqrt{x^2}$
Since $(x)^2 = x^2$ and we want the positive root.	$\lvert x\rvert$

ⓑ

	$\sqrt[3]{n^3}$
Since $(n)^3 = n^3$. It is an odd root so there is no need for an absolute value sign.	n

ⓒ

	$\sqrt[4]{p^4}$
Since $(p)^4 = p^4$ and we want the positive root.	$\lvert p\rvert$

ⓓ

	$\sqrt[5]{y^5}$
Since $(y)^5 = y^5$. It is an odd root so there is no need for an absolute value sign.	y

TRY IT : : 9.179 Simplify: ⓐ $\sqrt{b^2}$ ⓑ $\sqrt[3]{w^3}$ ⓒ $\sqrt[4]{m^4}$ ⓓ $\sqrt[5]{q^5}$.

TRY IT : : 9.180 Simplify: ⓐ $\sqrt{y^2}$ ⓑ $\sqrt[3]{p^3}$ ⓒ $\sqrt[4]{z^4}$ ⓓ $\sqrt[5]{q^5}$.

EXAMPLE 9.91

Simplify: ⓐ $\sqrt[3]{y^{18}}$ ⓑ $\sqrt[4]{z^8}$.

Solution

ⓐ

	$\sqrt[3]{y^{18}}$
Since $\left(y^6\right)^3 = y^{18}$.	$\sqrt[3]{\left(y^6\right)^3}$
	y^6

ⓑ

	$\sqrt[4]{z^8}$
Since $\left(z^2\right)^4 = z^8$.	$\sqrt[4]{\left(z^2\right)^4}$
Since z^2 is positive, we do not need an absolute value sign.	z^2

TRY IT : : 9.181 Simplify: ⓐ $\sqrt[4]{u^{12}}$ ⓑ $\sqrt[3]{v^{15}}$.

TRY IT : : 9.182 Simplify: ⓐ $\sqrt[5]{c^{20}}$ ⓑ $\sqrt[6]{d^{24}}$.

EXAMPLE 9.92

Simplify: ⓐ $\sqrt[3]{64p^6}$ ⓑ $\sqrt[4]{16q^{12}}$.

Solution

ⓐ

	$\sqrt[3]{64p^6}$
Rewrite $64p^6$ as $(4p^2)^3$.	$\sqrt[3]{(4p^2)^3}$
Take the cube root.	$4p^2$

ⓑ

	$\sqrt[4]{16q^{12}}$
Rewrite the radicand as a fourth power.	$\sqrt[4]{(2q^3)^4}$
Take the fourth root.	$2\lvert q^3\rvert$

TRY IT : : 9.183 Simplify: ⓐ $\sqrt[3]{27x^{27}}$ ⓑ $\sqrt[4]{81q^{28}}$.

TRY IT : : 9.184 Simplify: ⓐ $\sqrt[3]{125p^9}$ ⓑ $\sqrt[5]{243q^{25}}$.

Use the Product Property to Simplify Expressions with Higher Roots

We will simplify expressions with higher roots in much the same way as we simplified expressions with square roots. An *n*th root is considered simplified if it has no factors of m^n.

Simplified *n*th Root

$\sqrt[n]{a}$ is considered simplified if a has no factors of m^n.

We will generalize the Product Property of Square Roots to include any integer root $n \geq 2$.

Product Property of *n*th Roots

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \text{ and } \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

when $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and for any integer $n \geq 2$

EXAMPLE 9.93

Simplify: ⓐ $\sqrt[3]{x^4}$ ⓑ $\sqrt[4]{x^7}$.

Solution

ⓐ

	$\sqrt[3]{x^4}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{x^3 \cdot x}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{x^3} \cdot \sqrt[3]{x}$
Simplify.	$x\sqrt[3]{x}$

ⓑ

	$\sqrt[4]{x^7}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{x^4 \cdot x^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$
Simplify.	$\lvert x\rvert\sqrt[4]{x^3}$

TRY IT : : 9.185 Simplify: ⓐ $\sqrt[4]{y^6}$ ⓑ $\sqrt[3]{z^5}$.

TRY IT : : 9.186 Simplify: ⓐ $\sqrt[5]{p^8}$ ⓑ $\sqrt[6]{q^{13}}$.

EXAMPLE 9.94

Simplify: ⓐ $\sqrt[3]{16}$ ⓑ $\sqrt[4]{243}$.

Solution

ⓐ

	$\sqrt[3]{16}$
	$\sqrt[3]{2^4}$
Rewrite the radicand as a product using the greatest perfect cube factor.	$\sqrt[3]{2^3 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{2^3} \cdot \sqrt[3]{2}$
Simplify.	$2\sqrt[3]{2}$

ⓑ

	$\sqrt[4]{243}$
	$\sqrt[4]{3^5}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{3^4 \cdot 3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{3^4} \cdot \sqrt[4]{3}$
Simplify.	$3\sqrt[4]{3}$

TRY IT : : 9.187 Simplify: ⓐ $\sqrt[3]{81}$ ⓑ $\sqrt[4]{64}$.

TRY IT : : 9.188 Simplify: ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[4]{729}$.

Don’t forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

EXAMPLE 9.95

Simplify: ⓐ $\sqrt[3]{24x^7}$ ⓑ $\sqrt[4]{80y^{14}}$.

Solution

ⓐ

	$\sqrt[3]{24x^7}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{2^3x^6 \cdot 3x}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{2^3x^6} \cdot \sqrt[3]{3x}$
Rewrite the fir t radicand as $\left(2x^2\right)^3$.	$\sqrt[3]{\left(2x^2\right)^3} \cdot \sqrt[3]{3x}$
Simplify.	$2x^2\sqrt[3]{3x}$

ⓑ

	$\sqrt[4]{80y^{14}}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{2^4y^{12} \cdot 5y^2}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{2^4y^{12}} \cdot \sqrt[4]{5y^2}$
Rewrite the fir t radicand as $\left(2y^3\right)^4$.	$\sqrt[4]{\left(2y^3\right)^4} \cdot \sqrt[4]{5y^2}$
Simplify.	$2\left\lvert y^3\right\rvert\sqrt[4]{5y^2}$

TRY IT : : 9.189 Simplify: ⓐ $\sqrt[3]{54p^{10}}$ ⓑ $\sqrt[4]{64q^{10}}$.

TRY IT : : 9.190 Simplify: ⓐ $\sqrt[3]{128m^{11}}$ ⓑ $\sqrt[4]{162n^7}$.

EXAMPLE 9.96

Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-16}$.

Solution

ⓐ

	$\sqrt[3]{-27}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3}$
Take the cube root.	-3

ⓑ

	$\sqrt[4]{-16}$
There is no real number n where $n^4 = -16$.	Not a real number.

TRY IT : : 9.191 Simplify: ⓐ $\sqrt[3]{-108}$ ⓑ $\sqrt[4]{-48}$.

TRY IT : : 9.192 Simplify: ⓐ $\sqrt[3]{-625}$ ⓑ $\sqrt[4]{-324}$.

Use the Quotient Property to Simplify Expressions with Higher Roots

We can simplify higher roots with quotients in the same way we simplified square roots. First we simplify any fractions inside the radical.

EXAMPLE 9.97

Simplify: ⓐ $\sqrt[3]{\frac{a^8}{a^5}}$ ⓑ $\sqrt[4]{\frac{a^{10}}{a^2}}$.

Solution

ⓐ

	$\sqrt[3]{\frac{a^8}{a^5}}$
Simplify the fraction under the radical fir t.	$\sqrt[3]{a^3}$
Simplify.	a

ⓑ

	$\sqrt[4]{\frac{a^{10}}{a^2}}$
Simplify the fraction under the radical fir t.	$\sqrt[4]{a^8}$
Rewrite the radicand using perfect fourth power factors.	$\sqrt[4]{\left(a^2\right)^4}$
Simplify.	a^2

TRY IT : : 9.193 Simplify: ⓐ $\sqrt[4]{\frac{x^7}{x^3}}$ ⓑ $\sqrt[4]{\frac{y^{17}}{y^5}}$.

TRY IT : : 9.194 Simplify: ⓐ $\sqrt[3]{\frac{m^{13}}{m^7}}$ ⓑ $\sqrt[5]{\frac{n^{12}}{n^2}}$.

Previously, we used the Quotient Property 'in reverse' to simplify square roots. Now we will generalize the formula to include higher roots.

Quotient Property of *n*th Roots

$$\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$$

when $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0,$ and for any integer $n \geq 2$

EXAMPLE 9.98

Simplify: ⓐ $\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$ ⓑ $\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$.

Solution

ⓐ

	$\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$
Neither radicand is a perfect cube, so use the Quotient Property to write as one radical.	$\sqrt[3]{\frac{-108}{2}}$
Simplify the fraction under the radical.	$\sqrt[3]{-54}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{(-3)^3} \cdot \sqrt[3]{2}$
Simplify.	$-3\sqrt[3]{2}$

ⓑ

	$\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$
Neither radicand is a perfect fourth power, so use the Quotient Property to write as one radical.	$\sqrt[4]{\frac{96x^7}{3x^2}}$
Simplify the fraction under the radical.	$\sqrt[4]{32x^5}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{2^4 x^4 \cdot 2x}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{(2x)^4} \cdot \sqrt[4]{2x}$
Simplify.	$2\lvert x\rvert\sqrt[4]{2x}$

TRY IT : : 9.195

Simplify: ⓐ $\dfrac{\sqrt[3]{-532}}{\sqrt[3]{2}}$ ⓑ $\dfrac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$.

TRY IT : : 9.196

Simplify: ⓐ $\dfrac{\sqrt[3]{-192}}{\sqrt[3]{3}}$ ⓑ $\dfrac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$.

If the fraction inside the radical cannot be simplified, we use the first form of the Quotient Property to rewrite the expression as the quotient of two radicals.

EXAMPLE 9.99

Simplify: ⓐ $\sqrt[3]{\dfrac{24x^7}{y^3}}$ ⓑ $\sqrt[4]{\dfrac{48x^{10}}{y^8}}$.

Solution

ⓐ

	$\sqrt[3]{\dfrac{24x^7}{y^3}}$
The fraction in the radicand cannot be simplified. Use he Quotient Property to write as two radicals.	$\dfrac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$
Rewrite each radicand as a product using perfect cube factors.	$\dfrac{\sqrt[3]{8x^6 \cdot 3x}}{\sqrt[3]{y^3}}$
Rewrite the numerator as the product of two radicals.	$\dfrac{\sqrt[3]{(2x^2)^3}\,\sqrt[3]{3x}}{\sqrt[3]{y^3}}$
Simplify.	$\dfrac{2x^2\sqrt[3]{3x}}{y}$

ⓑ

$$\sqrt[4]{\frac{48x^{10}}{y^8}}$$

The fraction in the radicand cannot be simplified. Use he Quotient Property to write as two radicals.

$$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$$

Rewrite each radicand as a product using perfect fourth power factors.

$$\frac{\sqrt[4]{16x^8 \cdot 3x^2}}{\sqrt[4]{y^8}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[4]{\left(2x^2\right)^4}\sqrt[4]{3x^2}}{\sqrt[4]{\left(y^2\right)^4}}$$

Simplify.

$$\frac{2x^2\sqrt[4]{3x^2}}{y^2}$$

TRY IT : : 9.197 Simplify: ⓐ $\sqrt[3]{\frac{108c^{10}}{d^6}}$ ⓑ $\sqrt[4]{\frac{80x^{10}}{y^5}}$.

TRY IT : : 9.198 Simplify: ⓐ $\sqrt[3]{\frac{40r^3}{s}}$ ⓑ $\sqrt[4]{\frac{162m^{14}}{n^{12}}}$.

Add and Subtract Higher Roots

We can add and subtract higher roots like we added and subtracted square roots. First we provide a formal definition of like radicals.

Like Radicals

Radicals with the same index and same radicand are called **like radicals**.

Like radicals have the same index and the same radicand.

- $9\sqrt[4]{42x}$ and $-2\sqrt[4]{42x}$ are like radicals.
- $5\sqrt[3]{125x}$ and $6\sqrt[3]{125y}$ are not like radicals. The radicands are different.
- $2\sqrt[5]{1000q}$ and $-4\sqrt[4]{1000q}$ are not like radicals. The indices are different.

We add and subtract like radicals in the same way we add and subtract like terms. We can add $9\sqrt[4]{42x} + \left(-2\sqrt[4]{42x}\right)$ and the result is $7\sqrt[4]{42x}$.

EXAMPLE 9.100

Simplify: ⓐ $\sqrt[3]{4x} + \sqrt[3]{4x}$ ⓑ $4\sqrt[4]{8} - 2\sqrt[4]{8}$.

Solution

ⓐ

	$\sqrt[3]{4x}+\sqrt[3]{4x}$
The radicals are like, so we add the coefficient	$2\sqrt[3]{4x}$

ⓑ

	$4\sqrt[4]{8}-2\sqrt[4]{8}$
The radicals are like, so we subtract the coefficient	$2\sqrt[4]{8}$

TRY IT : : 9.199 Simplify: ⓐ $\sqrt[5]{3x}+\sqrt[5]{3x}$ ⓑ $3\sqrt[3]{9}-\sqrt[3]{9}$.

TRY IT : : 9.200 Simplify: ⓐ $\sqrt[4]{10y}+\sqrt[4]{10y}$ ⓑ $5\sqrt[6]{32}-3\sqrt[6]{32}$.

When an expression does not appear to have like radicals, we will simplify each radical first. Sometimes this leads to an expression with like radicals.

EXAMPLE 9.101

Simplify: ⓐ $\sqrt[3]{54}-\sqrt[3]{16}$ ⓑ $\sqrt[4]{48}+\sqrt[4]{243}$.

Solution

ⓐ

	$\sqrt[3]{54}-\sqrt[3]{16}$
Rewrite each radicand using perfect cube factors.	$\sqrt[3]{27}\cdot\sqrt[3]{2}-\sqrt[3]{8}\cdot\sqrt[3]{2}$
Rewrite the perfect cubes.	$\sqrt[3]{(3)^3}\sqrt[3]{2}-\sqrt[3]{(2)^3}\sqrt[3]{2}$
Simplify the radicals where possible.	$3\sqrt[3]{2}-2\sqrt[3]{2}$
Combine like radicals.	$\sqrt[3]{2}$

ⓑ

	$\sqrt[4]{48}+\sqrt[4]{243}$
Rewrite using perfect fourth power factors.	$\sqrt[4]{16}\cdot\sqrt[4]{3}+\sqrt[4]{81}\cdot\sqrt[4]{3}$
Rewrite the perfect fourth powers.	$\sqrt[4]{(2)^4}\sqrt[4]{3}+\sqrt[4]{(3)^4}\sqrt[4]{3}$
Simplify the radicals where possible.	$2\sqrt[4]{3}+3\sqrt[4]{3}$
Combine like radicals.	$5\sqrt[4]{3}$

TRY IT : : 9.201 Simplify: ⓐ $\sqrt[3]{192}-\sqrt[3]{81}$ ⓑ $\sqrt[4]{32}+\sqrt[4]{512}$.

TRY IT : : 9.202 Simplify: ⓐ $\sqrt[3]{108}-\sqrt[3]{250}$ ⓑ $\sqrt[5]{64}+\sqrt[5]{486}$.

EXAMPLE 9.102

Simplify: ⓐ $\sqrt[3]{24x^4}-\sqrt[3]{-81x^7}$ ⓑ $\sqrt[4]{162y^9}+\sqrt[4]{516y^5}$.

Solution

ⓐ

	$\sqrt[3]{24x^4}-\sqrt[3]{-81x^7}$
Rewrite each radicand using perfect cube factors.	$\sqrt[3]{8x^3}\cdot\sqrt[3]{3x}-\sqrt[3]{-27x^6}\cdot\sqrt[3]{3x}$
Rewrite the perfect cubes.	$\sqrt[3]{(2x)^3}\sqrt[3]{3x}-\sqrt[3]{\left(-3x^2\right)^3}\sqrt[3]{3x}$
Simplify the radicals where possible.	$2x\sqrt[3]{3x}-\left(-3x^2\sqrt[3]{3x}\right)$

ⓑ

	$\sqrt[4]{162y^9}+\sqrt[4]{516y^5}$
Rewrite each radicand using perfect fourth power factors.	$\sqrt[4]{81y^8}\cdot\sqrt[4]{2y}+\sqrt[4]{256y^4}\cdot\sqrt[4]{2y}$
Rewrite the perfect fourth powers.	$\sqrt[4]{\left(3y^2\right)^4}\cdot\sqrt[4]{2y}+\sqrt[4]{(4y)^4}\cdot\sqrt[4]{2y}$
Simplify the radicals where possible.	$3y^2\sqrt[4]{2y}+4\lvert y\rvert\sqrt[4]{2y}$

TRY IT : : 9.203 Simplify: ⓐ $\sqrt[3]{32y^5}-\sqrt[3]{-108y^8}$ ⓑ $\sqrt[4]{243r^{11}}+\sqrt[4]{768r^{10}}$.

TRY IT : : 9.204 Simplify: ⓐ $\sqrt[3]{40z^7}-\sqrt[3]{-135z^4}$ ⓑ $\sqrt[4]{80s^{13}}+\sqrt[4]{1280s^6}$.

MEDIA : :

Access these online resources for additional instruction and practice with simplifying higher roots.

- **Simplifying Higher Roots (https://openstax.org/l/25SimplifyHR)**
- **Add/Subtract Roots with Higher Indices (https://openstax.org/l/25AddSubtrHR)**

9.7 EXERCISES

Practice Makes Perfect

Simplify Expressions with Higher Roots

In the following exercises, simplify.

442.
ⓐ $\sqrt[3]{216}$
ⓑ $\sqrt[4]{256}$
ⓒ $\sqrt[5]{32}$

443.
ⓐ $\sqrt[3]{27}$
ⓑ $\sqrt[4]{16}$
ⓒ $\sqrt[5]{243}$

444.
ⓐ $\sqrt[3]{512}$
ⓑ $\sqrt[4]{81}$
ⓒ $\sqrt[5]{1}$

445.
ⓐ $\sqrt[3]{125}$
ⓑ $\sqrt[4]{1296}$
ⓒ $\sqrt[5]{1024}$

446.
ⓐ $\sqrt[3]{-8}$
ⓑ $\sqrt[4]{-81}$
ⓒ $\sqrt[5]{-32}$

447.
ⓐ $\sqrt[3]{-64}$
ⓑ $\sqrt[4]{-16}$
ⓒ $\sqrt[5]{-243}$

448.
ⓐ $\sqrt[3]{-125}$
ⓑ $\sqrt[4]{-1296}$
ⓒ $\sqrt[5]{-1024}$

449.
ⓐ $\sqrt[3]{-512}$
ⓑ $\sqrt[4]{-81}$
ⓒ $\sqrt[5]{-1}$

450.
ⓐ $\sqrt[5]{u^5}$
ⓑ $\sqrt[8]{v^8}$

451.
ⓐ $\sqrt[3]{a^3}$
ⓑ $\sqrt[12]{b^{12}}$

452.
ⓐ $\sqrt[4]{y^4}$
ⓑ $\sqrt[7]{m^7}$

453.
ⓐ $\sqrt[8]{k^8}$
ⓑ $\sqrt[6]{p^6}$

454.
ⓐ $\sqrt[3]{x^9}$
ⓑ $\sqrt[4]{y^{12}}$

455.
ⓐ $\sqrt[5]{a^{10}}$
ⓑ $\sqrt[3]{b^{27}}$

456.
ⓐ $\sqrt[4]{m^8}$
ⓑ $\sqrt[5]{n^{20}}$

457.
ⓐ $\sqrt[6]{r^{12}}$
ⓑ $\sqrt[3]{s^{30}}$

458.
ⓐ $\sqrt[4]{16x^8}$
ⓑ $\sqrt[6]{64y^{12}}$

459.
ⓐ $\sqrt[3]{-8c^9}$
ⓑ $\sqrt[3]{125d^{15}}$

460.
ⓐ $\sqrt[3]{216a^6}$
ⓑ $\sqrt[5]{32b^{20}}$

461.
ⓐ $\sqrt[7]{128r^{14}}$
ⓑ $\sqrt[4]{81s^{24}}$

Use the Product Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.

462. ⓐ $\sqrt[3]{r^5}$ ⓑ $\sqrt[4]{s^{10}}$

463. ⓐ $\sqrt[5]{u^7}$ ⓑ $\sqrt[6]{v^{11}}$

464. ⓐ $\sqrt[4]{m^5}$ ⓑ $\sqrt[8]{n^{10}}$

465. ⓐ $\sqrt[5]{p^8}$ ⓑ $\sqrt[3]{q^8}$

466. ⓐ $\sqrt[4]{32}$ ⓑ $\sqrt[5]{64}$

467. ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

468. ⓐ $\sqrt[5]{64}$ ⓑ $\sqrt[3]{256}$

469. ⓐ $\sqrt[4]{3125}$ ⓑ $\sqrt[3]{81}$

470. ⓐ $\sqrt[3]{108x^5}$ ⓑ $\sqrt[4]{48y^6}$

471. ⓐ $\sqrt[5]{96a^7}$ ⓑ $\sqrt[3]{375b^4}$

472. ⓐ $\sqrt[4]{405m^{10}}$ ⓑ $\sqrt[5]{160n^8}$

473. ⓐ $\sqrt[3]{512p^5}$ ⓑ $\sqrt[4]{324q^7}$

474. ⓐ $\sqrt[3]{-864}$ ⓑ $\sqrt[4]{-256}$

475. ⓐ $\sqrt[5]{-486}$ ⓑ $\sqrt[6]{-64}$

476. ⓐ $\sqrt[5]{-32}$ ⓑ $\sqrt[8]{-1}$

477. ⓐ $\sqrt[3]{-8}$ ⓑ $\sqrt[4]{-16}$

Use the Quotient Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.

478. ⓐ $\sqrt[3]{\frac{p^{11}}{p^2}}$ ⓑ $\sqrt[4]{\frac{q^{17}}{q^{13}}}$

479. ⓐ $\sqrt[5]{\frac{d^{12}}{d^7}}$ ⓑ $\sqrt[8]{\frac{m^{12}}{m^4}}$

480. ⓐ $\sqrt[5]{\frac{u^{21}}{u^{11}}}$ ⓑ $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

481. ⓐ $\sqrt[3]{\frac{r^{14}}{r^5}}$ ⓑ $\sqrt[4]{\frac{c^{21}}{c^9}}$

482. ⓐ $\frac{\sqrt[4]{64}}{\sqrt[4]{2}}$ ⓑ $\frac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

483. ⓐ $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$ ⓑ $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

484. ⓐ $\sqrt[3]{\frac{1050}{2}}$ ⓑ $\sqrt[4]{\frac{486y^9}{2y^3}}$

485. ⓐ $\sqrt[3]{\frac{162}{6}}$ ⓑ $\sqrt[4]{\frac{160r^{10}}{5r^3}}$

486. ⓐ $\sqrt[3]{\frac{54a^8}{b^3}}$ ⓑ $\sqrt[4]{\frac{64c^5}{d^2}}$

487. ⓐ $\sqrt[5]{\frac{96r^{11}}{s^3}}$ ⓑ $\sqrt[6]{\frac{128u^7}{v^3}}$

488. ⓐ $\sqrt[3]{\frac{81s^8}{t^3}}$ ⓑ $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$

489. ⓐ $\sqrt[3]{\frac{625u^{10}}{v^3}}$ ⓑ $\sqrt[4]{\frac{729c^{21}}{d^8}}$

Add and Subtract Higher Roots

In the following exercises, simplify.

490.
ⓐ $\sqrt[7]{8p}+\sqrt[7]{8p}$
ⓑ $3\sqrt[3]{25}-\sqrt[3]{25}$

491.
ⓐ $\sqrt[3]{15q}+\sqrt[3]{15q}$
ⓑ $2\sqrt[4]{27}-6\sqrt[4]{27}$

492.
ⓐ $3\sqrt[5]{9x}+7\sqrt[5]{9x}$
ⓑ $8\sqrt[7]{3q}-2\sqrt[7]{3q}$

493.
ⓐ $23\sqrt[13]{4y}+19\sqrt[13]{4y}$
ⓑ $31\sqrt[19]{5z}-17\sqrt[19]{5z}$

494.
ⓐ $\sqrt[3]{81}-\sqrt[3]{192}$
ⓑ $\sqrt[4]{512}-\sqrt[4]{32}$

495.
ⓐ $\sqrt[3]{250}-\sqrt[3]{54}$
ⓑ $\sqrt[4]{243}-\sqrt[4]{1875}$

496.
ⓐ $\sqrt[3]{128}+\sqrt[3]{250}$
ⓑ $\sqrt[5]{729}+\sqrt[5]{96}$

497.
ⓐ $\sqrt[4]{243}+\sqrt[4]{1250}$
ⓑ $\sqrt[3]{2000}+\sqrt[3]{54}$

498.
ⓐ $\sqrt[3]{64a^{10}}-\sqrt[3]{-216a^{12}}$
ⓑ $\sqrt[4]{486u^7}+\sqrt[4]{768u^3}$

499.
ⓐ $\sqrt[3]{80b^5}-\sqrt[3]{-270b^3}$
ⓑ $\sqrt[4]{160v^{10}}-\sqrt[4]{1280v^3}$

Mixed Practice

In the following exercises, simplify.

500. $\sqrt[4]{16}$

501. $\sqrt[6]{64}$

502. $\sqrt[3]{a^3}$

503. $\sqrt[12]{b^{12}}$

504. $\sqrt[3]{-8c^9}$

505. $\sqrt[3]{125d^{15}}$

506. $\sqrt[3]{r^5}$

507. $\sqrt[4]{s^{10}}$

508. $\sqrt[3]{108x^5}$

509. $\sqrt[4]{48y^6}$

510. $\sqrt[5]{-486}$

511. $\sqrt[6]{-64}$

512. $\dfrac{\sqrt[4]{64}}{\sqrt[4]{2}}$

513. $\dfrac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

514. $\sqrt[5]{\dfrac{96r^{11}}{s^3}}$

515. $\sqrt[6]{\dfrac{128u^7}{v^3}}$

516. $\sqrt[3]{81} - \sqrt[3]{192}$

517. $\sqrt[4]{512} - \sqrt[4]{32}$

518. $\sqrt[3]{64a^{10}} - \sqrt[3]{-216a^{12}}$

519. $\sqrt[4]{486u^7} + \sqrt[4]{768u^3}$

Everyday Math

520. Population growth The expression $10 \cdot x^n$ models the growth of a mold population after n generations. There were 10 spores at the start, and each had x offspring. So $10 \cdot x^n$ is the number of offspring at the fifth generation. At the fifth generation there were 10,240 offspring. Simplify the expression $\sqrt[5]{\dfrac{10{,}240}{10}}$ to determine the number of offspring of each spore.

521. Spread of a virus The expression $3 \cdot x^n$ models the spread of a virus after n cycles. There were three people originally infected with the virus, and each of them infected x people. So $3 \cdot x^4$ is the number of people infected on the fourth cycle. At the fourth cycle 1875 people were infected. Simplify the expression $\sqrt[4]{\dfrac{1875}{3}}$ to determine the number of people each person infected.

Writing Exercises

522. Explain how you know that $\sqrt[5]{x^{10}} = x^2$.

523. Explain why $\sqrt[4]{-64}$ is not a real number but $\sqrt[3]{-64}$ is.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can…	Confidently	With some help	No-I don't get it!
simplify expressions with higher roots.			
use the Product Property to simplify expressions with higher roots.			
use the Quotient Property to simplify expressions with higher roots.			
add and subtract higher roots.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

9.8 Rational Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the Laws of Exponents to simply expressions with rational exponents

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{15}+\frac{5}{12}$.
   If you missed this problem, review **Example 1.81**.
2. Simplify: $\left(4x^2y^5\right)^3$.
   If you missed this problem, review **Example 6.24**.
3. Simplify: 5^{-3}.
   If you missed this problem, review **Example 6.89**.

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use **rational exponents**, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(a^m)^n = a^{m\cdot n}$ when *m* and *n* are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number *p* such that $(8^p)^3 = 8$. We will use the Power Property of Exponents to find the value of *p*.

$$\begin{array}{lrcl} & (8^p)^3 & = & 8 \\ \text{Multiply the exponents on the left.} & 8^{3p} & = & 8 \\ \text{Write the exponent 1 on the right.} & 8^{3p} & = & 8^1 \\ \text{The exponents must be equal.} & 3p & = & 1 \\ \text{Solve for } p. & p & = & \frac{1}{3} \end{array}$$

$$\text{So}\left(8^{\frac{1}{3}}\right)^3 = 8.$$

But we know also $\left(\sqrt[3]{8}\right)^3 = 8$. Then it must be that $8^{\frac{1}{3}} = \sqrt[3]{8}$.

This same logic can be used for any positive integer exponent *n* to show that $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Rational Exponent $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ is a real number and $n \geq 2$, $a^{\frac{1}{n}} = \sqrt[n]{a}$.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

EXAMPLE 9.103

Write as a radical expression: ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$.

Solution

We want to write each expression in the form $\sqrt[n]{a}$.

ⓐ

	$x^{\frac{1}{2}}$
The denominator of the exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.	$\sqrt{x}$

ⓑ

	$y^{\frac{1}{3}}$
The denominator of the exponent is 3, so the index is 3.	$\sqrt[3]{y}$

ⓒ

	$z^{\frac{1}{4}}$
The denominator of the exponent is 4, so the index is 4.	$\sqrt[4]{z}$

TRY IT : : 9.205

Write as a radical expression: ⓐ $t^{\frac{1}{2}}$ ⓑ $m^{\frac{1}{3}}$ ⓒ $r^{\frac{1}{4}}$.

TRY IT : : 9.206

Write as a radial expression: ⓐ $b^{\frac{1}{2}}$ ⓑ $z^{\frac{1}{3}}$ ⓒ $p^{\frac{1}{4}}$.

EXAMPLE 9.104

Write with a rational exponent: ⓐ $\sqrt{x}$ ⓑ $\sqrt[3]{y}$ ⓒ $\sqrt[4]{z}$.

Solution

We want to write each radical in the form $a^{\frac{1}{n}}$.

ⓐ

	$\sqrt{x}$
No index is shown, so it is 2. The denominator of the exponent will be 2.	$x^{\frac{1}{2}}$

ⓑ

	$\sqrt[3]{y}$
The index is 3, so the denominator of the exponent is 3.	$y^{\frac{1}{3}}$

ⓒ

	$\sqrt[4]{z}$
The index is 4, so the denominator of the exponent is 4.	$z^{\frac{1}{4}}$

> **TRY IT : :** 9.207 Write with a rational exponent: ⓐ $\sqrt{s}$ ⓑ $\sqrt[3]{x}$ ⓒ $\sqrt[4]{b}$.

> **TRY IT : :** 9.208 Write with a rational exponent: ⓐ $\sqrt{v}$ ⓑ $\sqrt[3]{p}$ ⓒ $\sqrt[4]{p}$.

EXAMPLE 9.105

Write with a rational exponent: ⓐ $\sqrt{5y}$ ⓑ $\sqrt[3]{4x}$ ⓒ $3\sqrt[4]{5z}$.

Solution

We want to write each radical in the form $a^{\frac{1}{n}}$.

ⓐ

	$\sqrt{5y}$
No index is shown, so it is 2. The denominator of the exponent will be 2.	$(5y)^{\frac{1}{2}}$

ⓑ

	$\sqrt[3]{4x}$
The index is 3, so the denominator of the exponent is 3.	$(4x)^{\frac{1}{3}}$

ⓒ

	$3\sqrt[4]{5z}$
The index is 4, so the denominator of the exponent is 4.	$3(5z)^{\frac{1}{4}}$

> **TRY IT : :** 9.209 Write with a rational exponent: ⓐ $\sqrt{10m}$ ⓑ $\sqrt[5]{3n}$ ⓒ $3\sqrt[4]{6y}$.

> **TRY IT : :** 9.210 Write with a rational exponent: ⓐ $\sqrt[7]{3k}$ ⓑ $\sqrt[4]{5j}$ ⓒ $8\sqrt[3]{2a}$.

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

EXAMPLE 9.106

Simplify: ⓐ $25^{\frac{1}{2}}$ ⓑ $64^{\frac{1}{3}}$ ⓒ $256^{\frac{1}{4}}$.

Solution

ⓐ

	$25^{\frac{1}{2}}$
Rewrite as a square root.	$\sqrt{25}$
Simplify.	5

ⓑ

	$64^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{64}$
Recognize 64 is a perfect cube.	$\sqrt[3]{4^3}$
Simplify.	4

ⓒ

	$256^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{256}$
Recognize 256 is a perfect fourth power.	$\sqrt[4]{4^4}$
Simplify.	4

TRY IT : : 9.211

Simplify: ⓐ $36^{\frac{1}{2}}$ ⓑ $8^{\frac{1}{3}}$ ⓒ $16^{\frac{1}{4}}$.

TRY IT : : 9.212

Simplify: ⓐ $100^{\frac{1}{2}}$ ⓑ $27^{\frac{1}{3}}$ ⓒ $81^{\frac{1}{4}}$.

Be careful of the placement of the negative signs in the next example. We will need to use the property $a^{-n} = \frac{1}{a^n}$ in one case.

EXAMPLE 9.107

Simplify: ⓐ $(-64)^{\frac{1}{3}}$ ⓑ $-64^{\frac{1}{3}}$ ⓒ $(64)^{-\frac{1}{3}}$.

Solution

ⓐ

	$(-64)^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{-64}$
Rewrite -64 as a perfect cube.	$\sqrt[3]{(-4)^3}$
Simplify.	-4

ⓑ

	$-64^{\frac{1}{3}}$
The exponent applies only to the 64.	$-\left(64^{\frac{1}{3}}\right)$
Rewrite as a cube root.	$-\sqrt[3]{64}$
Rewrite 64 as 4^3.	$-\sqrt[3]{4^3}$
Simplify.	-4

ⓒ

	$(64)^{-\frac{1}{3}}$
Rewrite as a fraction with a positive exponent, using the property, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\sqrt[3]{64}}$
Write as a cube root. Rewrite 64 as 4^3.	$\frac{1}{\sqrt[3]{4^3}}$
Simplify.	$\frac{1}{4}$

TRY IT : : 9.213

Simplify: ⓐ $(-125)^{\frac{1}{3}}$ ⓑ $-125^{\frac{1}{3}}$ ⓒ $(125)^{-\frac{1}{3}}$.

TRY IT : : 9.214

Simplify: ⓐ $(-32)^{\frac{1}{5}}$ ⓑ $-32^{\frac{1}{5}}$ ⓒ $(32)^{-\frac{1}{5}}$.

EXAMPLE 9.108

Simplify: ⓐ $(-16)^{\frac{1}{4}}$ ⓑ $-16^{\frac{1}{4}}$ ⓒ $(16)^{-\frac{1}{4}}$.

Solution

ⓐ

	$(-16)^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{-16}$
There is no real number whose fourth power is -16.	

ⓑ

	$-16^{\frac{1}{4}}$
The exponent only applies to the 16. Rewrite as a fourth root.	$-\sqrt[4]{16}$
Rewrite 16 as 2^4.	$-\sqrt[4]{2^4}$
Simplify.	-2

ⓒ

	$(16)^{-\frac{1}{4}}$
Rewrite using the property $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{(16)^{\frac{1}{4}}}$
Rewrite as a fourth root.	$\frac{1}{\sqrt[4]{16}}$
Rewrite 16 as 2^4.	$\frac{1}{\sqrt[4]{2^4}}$
Simplify.	$\frac{1}{2}$

> **TRY IT : :** 9.215
>
> Simplify: ⓐ $(-64)^{\frac{1}{2}}$ ⓑ $-64^{\frac{1}{2}}$ ⓒ $(64)^{-\frac{1}{2}}$.

> **TRY IT : :** 9.216
>
> Simplify: ⓐ $(-256)^{\frac{1}{4}}$ ⓑ $-256^{\frac{1}{4}}$ ⓒ $(256)^{-\frac{1}{4}}$.

Simplify Expressions with $a^{\frac{m}{n}}$

Let's work with the Power Property for Exponents some more.

Suppose we raise $a^{\frac{1}{n}}$ to the power *m*.

	$\left(a^{\frac{1}{n}}\right)^m$
Multiply the exponents.	$a^{\frac{1}{n} \cdot m}$
Simplify.	$a^{\frac{m}{n}}$

So $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$.

Now suppose we take a^m to the $\frac{1}{n}$ power.

	$(a^m)^{\frac{1}{n}}$
Multiply the exponents.	$a^{m \cdot \frac{1}{n}}$
Simplify.	$a^{\frac{m}{n}}$

So $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ also.

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller.

Rational Exponent $a^{\frac{m}{n}}$

For any positive integers *m* and *n*,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$
$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

EXAMPLE 9.109

Write with a rational exponent: ⓐ $\sqrt{y^3}$ ⓑ $\sqrt[3]{x^2}$ ⓒ $\sqrt[4]{z^3}$.

Solution

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$.

ⓐ

The numerator of the exponent is the exponent of y, 3.	$\sqrt[2]{y^3}$
The denominator of the exponent is the index of the radical, 2.	$y^{\frac{3}{2}}$

ⓑ

The numerator of the exponent is the exponent of x, 2.	$\sqrt[3]{x^2}$
The denominator of the exponent is the index of the radical, 3.	$x^{\frac{2}{3}}$

ⓒ

The numerator of the exponent is the exponent of z, 3.	$\sqrt[4]{z^3}$
The denominator of the exponent is the index of the radical, 4.	$z^{\frac{3}{4}}$

TRY IT : : 9.217 Write with a rational exponent: ⓐ $\sqrt{x^5}$ ⓑ $\sqrt[4]{z^3}$ ⓒ $\sqrt[5]{y^2}$.

TRY IT : : 9.218 Write with a rational exponent: ⓐ $\sqrt[5]{a^2}$ ⓑ $\sqrt[3]{b^7}$ ⓒ $\sqrt[4]{m^5}$.

EXAMPLE 9.110

Simplify: ⓐ $9^{\frac{3}{2}}$ ⓑ $125^{\frac{2}{3}}$ ⓒ $81^{\frac{3}{4}}$.

Solution

We will rewrite each expression as a radical first using the property, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

ⓐ

	$9^{\frac{3}{2}}$
The power of the radical is the numerator of the exponent, 3. Since the denominator of the exponent is 2, this is a square root.	$(\sqrt{9})^3$
Simplify.	$(3)^3$
	27

ⓑ

	$125^{\frac{2}{3}}$
The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.	$\left(\sqrt[3]{125}\right)^2$
Simplify.	$(5)^2$
	25

ⓒ

	$81^{\frac{3}{4}}$
The power of the radical is the numerator of the exponent, 3. The index of the radical is the denominator of the exponent, 4.	$\left(\sqrt[4]{81}\right)^3$
Simplify.	$(3)^3$
	27

TRY IT : : 9.219 Simplify: ⓐ $4^{\frac{3}{2}}$ ⓑ $27^{\frac{2}{3}}$ ⓒ $625^{\frac{3}{4}}$.

TRY IT : : 9.220 Simplify: ⓐ $8^{\frac{5}{3}}$ ⓑ $81^{\frac{3}{2}}$ ⓒ $16^{\frac{3}{4}}$.

Remember that $b^{-p} = \frac{1}{b^p}$. The negative sign in the exponent does not change the sign of the expression.

EXAMPLE 9.111

Simplify: ⓐ $16^{-\frac{3}{2}}$ ⓑ $32^{-\frac{2}{5}}$ ⓒ $4^{-\frac{5}{2}}$.

Solution

We will rewrite each expression first using $b^{-p} = \frac{1}{b^p}$ and then change to radical form.

ⓐ

	$16^{-\frac{3}{2}}$
Rewrite using $b^{-p} = \frac{1}{b^p}$.	$\frac{1}{16^{\frac{3}{2}}}$
Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.	$\frac{1}{\left(\sqrt{16}\right)^3}$
Simplify.	$\frac{1}{4^3}$
	$\frac{1}{64}$

ⓑ

	$32^{-\frac{2}{5}}$
Rewrite using $b^{-p} = \frac{1}{b^p}$.	$\frac{1}{32^{\frac{2}{5}}}$
Change to radical form.	$\frac{1}{\left(\sqrt[5]{32}\right)^2}$
Rewrite the radicand as a power.	$\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$
Simplify.	$\frac{1}{2^2}$
	$\frac{1}{4}$

ⓒ

	$4^{-\frac{5}{2}}$
Rewrite using $b^{-p} = \frac{1}{b^p}$.	$\frac{1}{4^{\frac{5}{2}}}$
Change to radical form.	$\frac{1}{(\sqrt{4})^5}$
Simplify.	$\frac{1}{2^5}$
	$\frac{1}{32}$

TRY IT : : 9.221

Simplify: ⓐ $8^{-\frac{5}{3}}$ ⓑ $81^{-\frac{3}{2}}$ ⓒ $16^{-\frac{3}{4}}$.

TRY IT : : 9.222

Simplify: ⓐ $4^{-\frac{3}{2}}$ ⓑ $27^{-\frac{2}{3}}$ ⓒ $625^{-\frac{3}{4}}$.

EXAMPLE 9.112

Simplify: ⓐ $-25^{\frac{3}{2}}$ ⓑ $-25^{-\frac{3}{2}}$ ⓒ $(-25)^{\frac{3}{2}}$.

Solution

ⓐ

	$-25^{\frac{3}{2}}$
Rewrite in radical form.	$-(\sqrt{25})^3$
Simplify the radical.	$-(5)^3$
Simplify.	-125

ⓑ

	$-25^{-\frac{3}{2}}$
Rewrite using $b^{-p} = \frac{1}{b^p}$.	$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$
Rewrite in radical form.	$-\left(\frac{1}{(\sqrt{25})^3}\right)$
Simplify the radical.	$-\left(\frac{1}{(5)^3}\right)$
Simplify.	$-\frac{1}{125}$

ⓒ

	$(-25)^{\frac{3}{2}}$
Rewrite in radical form.	$(\sqrt{-25})^3$
There is no real number whose square root is -25.	Not a real number.

TRY IT : : 9.223

Simplify: ⓐ $-16^{\frac{3}{2}}$ ⓑ $-16^{-\frac{3}{2}}$ ⓒ $(-16)^{-\frac{3}{2}}$.

TRY IT : : 9.224

Simplify: ⓐ $-81^{\frac{3}{2}}$ ⓑ $-81^{-\frac{3}{2}}$ ⓒ $(-81)^{-\frac{3}{2}}$.

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

The same laws of exponents that we already used apply to rational exponents, too. We will list the Exponent Properties here to have them for reference as we simplify expressions.

Summary of Exponent Properties

If a, b are real numbers and m, n are rational numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0, \quad m > n$
	$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad a \neq 0, \quad n > m$
Zero Exponent Definitio	$a^0 = 1, \quad a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \quad b \neq 0$

When we multiply the same base, we add the exponents.

EXAMPLE 9.113

Simplify: ⓐ $2^{\frac{1}{2}} \cdot 2^{\frac{5}{2}}$ ⓑ $x^{\frac{2}{3}} \cdot x^{\frac{4}{3}}$ ⓒ $z^{\frac{3}{4}} \cdot z^{\frac{5}{4}}$.

Solution

ⓐ

	$2^{\frac{1}{2}} \cdot 2^{\frac{5}{2}}$
The bases are the same, so we add the exponents.	$2^{\frac{1}{2}+\frac{5}{2}}$
Add the fractions.	$2^{\frac{6}{2}}$
Simplify the exponent.	2^3
Simplify.	8

ⓑ

	$x^{\frac{2}{3}} \cdot x^{\frac{4}{3}}$
The bases are the same, so we add the exponents.	$x^{\frac{2}{3}+\frac{4}{3}}$
Add the fractions.	$x^{\frac{6}{3}}$
Simplify.	x^2

ⓒ

	$z^{\frac{3}{4}} \cdot z^{\frac{5}{4}}$
The bases are the same, so we add the exponents.	$z^{\frac{3}{4}+\frac{5}{4}}$
Add the fractions.	$z^{\frac{8}{4}}$
Simplify.	z^2

TRY IT : : 9.225

Simplify: ⓐ $3^{\frac{2}{3}} \cdot 3^{\frac{4}{3}}$ ⓑ $y^{\frac{1}{3}} \cdot y^{\frac{8}{3}}$ ⓒ $m^{\frac{1}{4}} \cdot m^{\frac{3}{4}}$.

TRY IT : : 9.226

Simplify: ⓐ $5^{\frac{3}{5}} \cdot 5^{\frac{7}{5}}$ ⓑ $z^{\frac{1}{8}} \cdot z^{\frac{7}{8}}$ ⓒ $n^{\frac{2}{7}} \cdot n^{\frac{5}{7}}$.

We will use the Power Property in the next example.

EXAMPLE 9.114

Simplify: ⓐ $\left(x^4\right)^{\frac{1}{2}}$ ⓑ $\left(y^6\right)^{\frac{1}{3}}$ ⓒ $\left(z^9\right)^{\frac{2}{3}}$.

Solution

ⓐ

	$\left(x^4\right)^{\frac{1}{2}}$
To raise a power to a power, we multiply the exponents.	$x^{4 \cdot \frac{1}{2}}$
Simplify.	x^2

ⓑ

	$\left(y^6\right)^{\frac{1}{3}}$
To raise a power to a power, we multiply the exponents.	$y^{6 \cdot \frac{1}{3}}$
Simplify.	y^2

ⓒ

	$\left(z^9\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$z^{9 \cdot \frac{2}{3}}$
Simplify.	z^6

TRY IT : : 9.227

Simplify: ⓐ $\left(p^{10}\right)^{\frac{1}{5}}$ ⓑ $\left(q^8\right)^{\frac{3}{4}}$ ⓒ $\left(x^6\right)^{\frac{4}{3}}$.

TRY IT : : 9.228

Simplify: ⓐ $\left(r^6\right)^{\frac{5}{3}}$ ⓑ $\left(s^{12}\right)^{\frac{3}{4}}$ ⓒ $\left(m^9\right)^{\frac{2}{9}}$.

The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

EXAMPLE 9.115

Simplify: ⓐ $\dfrac{x^{\frac{4}{3}}}{x^{\frac{1}{3}}}$ ⓑ $\dfrac{y^{\frac{3}{4}}}{y^{\frac{1}{4}}}$ ⓒ $\dfrac{z^{\frac{2}{3}}}{z^{\frac{5}{3}}}$.

Solution

ⓐ

$$\frac{x^{\frac{4}{3}}}{x^{\frac{1}{3}}}$$

To divide with the same base, we subtract the exponents.	$x^{\frac{4}{3}-\frac{1}{3}}$
Simplify.	x

ⓑ

$$\frac{y^{\frac{3}{4}}}{y^{\frac{1}{4}}}$$

To divide with the same base, we subtract the exponents.	$y^{\frac{3}{4}-\frac{1}{4}}$
Simplify.	$y^{\frac{1}{2}}$

ⓒ

$$\frac{z^{\frac{2}{3}}}{z^{\frac{5}{3}}}$$

To divide with the same base, we subtract the exponents.	$z^{\frac{2}{3}-\frac{5}{3}}$
Rewrite without a negative exponent.	$\frac{1}{z}$

TRY IT : : 9.229

Simplify: ⓐ $\frac{u^{\frac{5}{4}}}{u^{\frac{1}{4}}}$ ⓑ $\frac{v^{\frac{3}{5}}}{v^{\frac{2}{5}}}$ ⓒ $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$.

TRY IT : : 9.230

Simplify: ⓐ $\frac{c^{\frac{12}{5}}}{c^{\frac{2}{5}}}$ ⓑ $\frac{m^{\frac{5}{4}}}{m^{\frac{9}{4}}}$ ⓒ $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$.

Sometimes we need to use more than one property. In the next two examples, we will use both the Product to a Power Property and then the Power Property.

EXAMPLE 9.116

Simplify: ⓐ $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ ⓑ $\left(8v^{\frac{1}{4}}\right)^{\frac{2}{3}}$.

Solution

ⓐ

	$\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
First we use the Product to a Power Property.	$(27)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
Rewrite 27 as a power of 3.	$\left(3^3\right)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$\left(3^2\right)\left(u^{\frac{1}{3}}\right)$
Simplify.	$9u^{\frac{1}{3}}$

ⓑ

	$\left(8v^{\frac{1}{4}}\right)^{\frac{2}{3}}$
First we use the Product to a Power Property.	$(8)^{\frac{2}{3}}\left(v^{\frac{1}{4}}\right)^{\frac{2}{3}}$
Rewrite 8 as a power of 2.	$\left(2^3\right)^{\frac{2}{3}}\left(v^{\frac{1}{4}}\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$\left(2^2\right)\left(v^{\frac{1}{6}}\right)$
Simplify.	$4v^{\frac{1}{6}}$

TRY IT : : 9.231

Simplify: ⓐ $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ ⓑ $\left(64y^{\frac{2}{3}}\right)^{\frac{1}{3}}$.

TRY IT : : 9.232

Simplify: ⓐ $\left(16m^{\frac{1}{3}}\right)^{\frac{3}{2}}$ ⓑ $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$.

EXAMPLE 9.117

Simplify: ⓐ $\left(m^3 n^9\right)^{\frac{1}{3}}$ ⓑ $\left(p^4 q^8\right)^{\frac{1}{4}}$.

Solution

ⓐ

	$\left(m^3 n^9\right)^{\frac{1}{3}}$
First we use the Product to a Power Property.	$\left(m^3\right)^{\frac{1}{3}}\left(n^9\right)^{\frac{1}{3}}$
To raise a power to a power, we multiply the exponents.	mn^3

ⓑ

	$\left(p^4 q^8\right)^{\frac{1}{4}}$
First we use the Product to a Power Property.	$\left(p^4\right)^{\frac{1}{4}}\left(q^8\right)^{\frac{1}{4}}$
To raise a power to a power, we multiply the exponents.	pq^2

We will use both the Product and Quotient Properties in the next example.

EXAMPLE 9.118

Simplify: ⓐ $\dfrac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ ⓑ $\dfrac{y^{\frac{4}{3}} \cdot y}{y^{-\frac{2}{3}}}$.

Solution

ⓐ

$$\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$$

Use the Product Property in the numerator, add the exponents. $\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$

Use the Quotient Property, subtract the exponents. $x^{\frac{8}{4}}$

Simplify. x^2

ⓑ

$$\frac{y^{\frac{4}{3}} \cdot y}{y^{-\frac{2}{3}}}$$

Use the Product Property in the numerator, add the exponents. $\frac{y^{\frac{7}{3}}}{y^{-\frac{2}{3}}}$

Use the Quotient Property, subtract the exponents. $y^{\frac{9}{3}}$

Simplify. y^3

TRY IT : : 9.233 Simplify: ⓐ $\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$ ⓑ $\frac{n^{\frac{1}{6}} \cdot n}{n^{-\frac{11}{6}}}$.

TRY IT : : 9.234 Simplify: ⓐ $\frac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$ ⓑ $\frac{v^{\frac{1}{2}} \cdot v}{v^{-\frac{7}{2}}}$.

9.8 EXERCISES

Practice Makes Perfect

Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

524.
ⓐ $x^{\frac{1}{2}}$
ⓑ $y^{\frac{1}{3}}$
ⓒ $z^{\frac{1}{4}}$

525.
ⓐ $r^{\frac{1}{2}}$
ⓑ $s^{\frac{1}{3}}$
ⓒ $t^{\frac{1}{4}}$

526.
ⓐ $u^{\frac{1}{5}}$
ⓑ $v^{\frac{1}{9}}$
ⓒ $w^{\frac{1}{20}}$

527.
ⓐ $g^{\frac{1}{7}}$
ⓑ $h^{\frac{1}{5}}$
ⓒ $j^{\frac{1}{25}}$

In the following exercises, write with a rational exponent.

528.
ⓐ $-\sqrt[7]{x}$
ⓑ $\sqrt[9]{y}$
ⓒ $\sqrt[5]{f}$

529.
ⓐ $\sqrt[8]{r}$
ⓑ $\sqrt[10]{s}$
ⓒ $\sqrt[4]{t}$

530.
ⓐ $\sqrt[3]{a}$
ⓑ $\sqrt[12]{b}$
ⓒ $\sqrt{c}$

531.
ⓐ $\sqrt[5]{u}$
ⓑ $\sqrt{v}$
ⓒ $\sqrt[16]{w}$

532.
ⓐ $\sqrt[3]{7c}$
ⓑ $\sqrt[7]{12d}$
ⓒ $3\sqrt[4]{5f}$

533.
ⓐ $\sqrt[4]{5x}$
ⓑ $\sqrt[8]{9y}$
ⓒ $7\sqrt[5]{3z}$

534.
ⓐ $\sqrt{21p}$
ⓑ $\sqrt[4]{8q}$
ⓒ $4\sqrt[6]{36r}$

535.
ⓐ $\sqrt[3]{25a}$
ⓑ $\sqrt{3b}$
ⓒ $\sqrt[10]{40c}$

In the following exercises, simplify.

536.
ⓐ $81^{\frac{1}{2}}$
ⓑ $125^{\frac{1}{3}}$
ⓒ $64^{\frac{1}{2}}$

537.
ⓐ $625^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$
ⓒ $32^{\frac{1}{5}}$

538.
ⓐ $16^{\frac{1}{4}}$
ⓑ $16^{\frac{1}{2}}$
ⓒ $3125^{\frac{1}{5}}$

539.
ⓐ $216^{\frac{1}{3}}$
ⓑ $32^{\frac{1}{5}}$
ⓒ $81^{\frac{1}{4}}$

540.
ⓐ $(-216)^{\frac{1}{3}}$
ⓑ $-216^{\frac{1}{3}}$
ⓒ $(216)^{-\frac{1}{3}}$

541.
ⓐ $(-243)^{\frac{1}{5}}$
ⓑ $-243^{\frac{1}{5}}$
ⓒ $(243)^{-\frac{1}{5}}$

542.
ⓐ $(-1)^{\frac{1}{3}}$
ⓑ $-1^{\frac{1}{3}}$
ⓒ $(1)^{-\frac{1}{3}}$

543.
ⓐ $(-1000)^{\frac{1}{3}}$
ⓑ $-1000^{\frac{1}{3}}$
ⓒ $(1000)^{-\frac{1}{3}}$

544.
ⓐ $(-81)^{\frac{1}{4}}$
ⓑ $-81^{\frac{1}{4}}$
ⓒ $(81)^{-\frac{1}{4}}$

545.
ⓐ $(-49)^{\frac{1}{2}}$
ⓑ $-49^{\frac{1}{2}}$
ⓒ $(49)^{-\frac{1}{2}}$

546.
ⓐ $(-36)^{\frac{1}{2}}$
ⓑ $-36^{\frac{1}{2}}$
ⓒ $(36)^{-\frac{1}{2}}$

547.
ⓐ $(-1)^{\frac{1}{4}}$
ⓑ $(1)^{-\frac{1}{4}}$
ⓒ $-1^{\frac{1}{4}}$

548.
ⓐ $(-100)^{\frac{1}{2}}$
ⓑ $-100^{\frac{1}{2}}$
ⓒ $(100)^{-\frac{1}{2}}$

549.
ⓐ $(-32)^{\frac{1}{5}}$
ⓑ $(243)^{-\frac{1}{5}}$
ⓒ $-125^{\frac{1}{3}}$

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

550.
ⓐ $\sqrt{m^5}$
ⓑ $\sqrt[3]{n^2}$
ⓒ $\sqrt[4]{p^3}$

551.
ⓐ $\sqrt[4]{r^7}$
ⓑ $\sqrt[5]{s^3}$
ⓒ $\sqrt[3]{t^7}$

552.
ⓐ $\sqrt[5]{u^2}$
ⓑ $\sqrt[5]{v^8}$
ⓒ $\sqrt[9]{w^4}$

553.
ⓐ $\sqrt[3]{a}$
ⓑ $\sqrt{b^5}$
ⓒ $\sqrt[3]{c^5}$

In the following exercises, simplify.

554.

ⓐ $16^{\frac{3}{2}}$

ⓑ $8^{\frac{2}{3}}$

ⓒ $10{,}000^{\frac{3}{4}}$

555.

ⓐ $1000^{\frac{2}{3}}$

ⓑ $25^{\frac{3}{2}}$

ⓒ $32^{\frac{3}{5}}$

556.

ⓐ $27^{\frac{5}{3}}$

ⓑ $16^{\frac{5}{4}}$

ⓒ $32^{\frac{2}{5}}$

557.

ⓐ $16^{\frac{3}{2}}$

ⓑ $125^{\frac{5}{3}}$

ⓒ $64^{\frac{4}{3}}$

558.

ⓐ $32^{\frac{2}{5}}$

ⓑ $27^{-\frac{2}{3}}$

ⓒ $25^{-\frac{3}{2}}$

559.

ⓐ $64^{\frac{5}{2}}$

ⓑ $81^{-\frac{3}{2}}$

ⓒ $27^{-\frac{4}{3}}$

560.

ⓐ $25^{\frac{3}{2}}$

ⓑ $9^{-\frac{3}{2}}$

ⓒ $(-64)^{\frac{2}{3}}$

561.

ⓐ $100^{\frac{3}{2}}$

ⓑ $49^{-\frac{5}{2}}$

ⓒ $(-100)^{\frac{3}{2}}$

562.

ⓐ $-9^{\frac{3}{2}}$

ⓑ $-9^{-\frac{3}{2}}$

ⓒ $(-9)^{\frac{3}{2}}$

563.

ⓐ $-64^{\frac{3}{2}}$

ⓑ $-64^{-\frac{3}{2}}$

ⓒ $(-64)^{\frac{3}{2}}$

564.

ⓐ $-100^{\frac{3}{2}}$

ⓑ $-100^{-\frac{3}{2}}$

ⓒ $(-100)^{\frac{3}{2}}$

565.

ⓐ $-49^{\frac{3}{2}}$

ⓑ $-49^{-\frac{3}{2}}$

ⓒ $(-49)^{\frac{3}{2}}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

566.

ⓐ $4^{\frac{5}{8}} \cdot 4^{\frac{11}{8}}$

ⓑ $m^{\frac{7}{12}} \cdot m^{\frac{17}{12}}$

ⓒ $p^{\frac{3}{7}} \cdot p^{\frac{18}{7}}$

567.

ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$

ⓑ $n^{\frac{2}{10}} \cdot n^{\frac{8}{10}}$

ⓒ $q^{\frac{2}{5}} \cdot q^{\frac{13}{5}}$

568.

ⓐ $5^{\frac{1}{2}} \cdot 5^{\frac{7}{2}}$

ⓑ $c^{\frac{3}{4}} \cdot c^{\frac{9}{4}}$

ⓒ $d^{\frac{3}{5}} \cdot d^{\frac{2}{5}}$

569.

ⓐ $10^{\frac{1}{3}} \cdot 10^{\frac{5}{3}}$

ⓑ $x^{\frac{5}{6}} \cdot x^{\frac{7}{6}}$

ⓒ $y^{\frac{11}{8}} \cdot y^{\frac{21}{8}}$

570.

ⓐ $\left(m^6\right)^{\frac{5}{2}}$

ⓑ $\left(n^9\right)^{\frac{4}{3}}$

ⓒ $\left(p^{12}\right)^{\frac{3}{4}}$

571.

ⓐ $\left(a^{12}\right)^{\frac{1}{6}}$

ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$

ⓒ $\left(c^{11}\right)^{\frac{1}{11}}$

572.

ⓐ $\left(x^{12}\right)^{\frac{2}{3}}$

ⓑ $\left(y^{20}\right)^{\frac{2}{5}}$

ⓒ $\left(z^{16}\right)^{\frac{1}{16}}$

573.

ⓐ $\left(h^{6}\right)^{\frac{4}{3}}$

ⓑ $\left(k^{12}\right)^{\frac{3}{4}}$

ⓒ $\left(j^{10}\right)^{\frac{7}{5}}$

574.

ⓐ $\dfrac{x^{\frac{7}{2}}}{x^{\frac{5}{2}}}$

ⓑ $\dfrac{y^{\frac{5}{2}}}{y^{\frac{1}{2}}}$

ⓒ $\dfrac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$

575.

ⓐ $\dfrac{s^{\frac{11}{5}}}{s^{\frac{6}{5}}}$

ⓑ $\dfrac{z^{\frac{7}{3}}}{z^{\frac{1}{3}}}$

ⓒ $\dfrac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

576.

ⓐ $\dfrac{t^{\frac{12}{5}}}{t^{\frac{7}{5}}}$

ⓑ $\dfrac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}}$

ⓒ $\dfrac{m^{\frac{13}{8}}}{m^{\frac{5}{8}}}$

577.

ⓐ $\dfrac{u^{\frac{13}{9}}}{u^{\frac{4}{9}}}$

ⓑ $\dfrac{r^{\frac{15}{7}}}{r^{\frac{8}{7}}}$

ⓒ $\dfrac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$

578.

ⓐ $\left(9p^{\frac{2}{3}}\right)^{\frac{5}{2}}$

ⓑ $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$

579.

ⓐ $\left(81r^{\frac{4}{5}}\right)^{\frac{1}{4}}$

ⓑ $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$

580.

ⓐ $\left(16u^{\frac{1}{3}}\right)^{\frac{3}{4}}$

ⓑ $\left(100v^{\frac{2}{5}}\right)^{\frac{3}{2}}$

581.

ⓐ $\left(27m^{\frac{3}{4}}\right)^{\frac{2}{3}}$

ⓑ $\left(625n^{\frac{8}{3}}\right)^{\frac{3}{4}}$

582.

ⓐ $\left(x^{8}y^{10}\right)^{\frac{1}{2}}$

ⓑ $\left(a^{9}b^{12}\right)^{\frac{1}{3}}$

583.

ⓐ $\left(r^{8}s^{4}\right)^{\frac{1}{4}}$

ⓑ $\left(u^{15}v^{20}\right)^{\frac{1}{5}}$

584.

ⓐ $\left(a^{6}b^{16}\right)^{\frac{1}{2}}$

ⓑ $\left(j^{9}k^{6}\right)^{\frac{2}{3}}$

585.

ⓐ $\left(r^{16}s^{10}\right)^{\frac{1}{2}}$

ⓑ $\left(u^{10}v^{5}\right)^{\frac{4}{5}}$

586.

ⓐ $\dfrac{r^{\frac{5}{2}}\cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$

ⓑ $\dfrac{s^{\frac{1}{5}}\cdot s}{s^{-\frac{9}{5}}}$

587.

ⓐ $\dfrac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$

ⓑ $\dfrac{b^{\frac{2}{3}} \cdot b}{b^{-\frac{7}{3}}}$

588.

ⓐ $\dfrac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$

ⓑ $\dfrac{d^{\frac{3}{5}} \cdot d}{d^{-\frac{2}{5}}}$

589.

ⓐ $\dfrac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$

ⓑ $\dfrac{n^{\frac{3}{7}} \cdot n}{n^{-\frac{4}{7}}}$

590. $4^{\frac{5}{2}} \cdot 4^{\frac{1}{2}}$

591. $n^{\frac{2}{6}} \cdot n^{\frac{4}{6}}$

592. $\left(a^{24}\right)^{\frac{1}{6}}$

593. $\left(b^{10}\right)^{\frac{3}{5}}$

594. $\dfrac{w^{\frac{2}{5}}}{w^{\frac{7}{5}}}$

595. $\dfrac{z^{\frac{2}{3}}}{z^{\frac{8}{3}}}$

596. $\left(27r^{\frac{3}{5}}\right)^{\frac{1}{3}}$

597. $\left(64s^{\frac{3}{5}}\right)^{\frac{1}{6}}$

598. $\left(r^9 s^{12}\right)^{\frac{1}{3}}$

599. $\left(u^{12} v^{18}\right)^{\frac{1}{6}}$

Everyday Math

600. Landscaping Joe wants to have a square garden plot in his backyard. He has enough compost to cover an area of 144 square feet. Simplify $144^{\frac{1}{2}}$ to find the length of each side of his garden.

601. Landscaping Elliott wants to make a square patio in his yard. He has enough concrete to pave an area of 242 square feet. Simplify $242^{\frac{1}{2}}$ to find the length of each side of his patio.Round to the nearest tenth of a foot.

602. Gravity While putting up holiday decorations, Bob dropped a decoration from the top of a tree that is 12 feet tall. Simplify $\dfrac{12^{\frac{1}{2}}}{16^{\frac{1}{2}}}$ to find how many seconds it took for the decoration to reach the ground. Round to the nearest tenth of a second.

603. Gravity An airplane dropped a flare from a height of 1024 feet above a lake. Simplify $\dfrac{1024^{\frac{1}{2}}}{16^{\frac{1}{2}}}$ to find how many seconds it took for the flare to reach the water.

Writing Exercises

604. Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.

605. Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

CHAPTER 9 REVIEW

KEY TERMS

index $\sqrt[n]{a}$ *n* is called the *index* of the radical.

like radicals Radicals with the same index and same radicand are called like radicals.

like square roots Square roots with the same radicand are called like square roots.

***n*th root of a number** If $b^n = a$, then b is an *n*th root of a.

principal *n*th root The principal *n*th root of a is written $\sqrt[n]{a}$.

radical equation An equation in which the variable is in the radicand of a square root is called a radical equation

rational exponents

- If $\sqrt[n]{a}$ is a real number and $n \geq 2$, $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- For any positive integers *m* and *n*, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

rationalizing the denominator The process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer is called rationalizing the denominator.

square of a number

- If $n^2 = m$, then m is the square of n

square root notation

- If $m = n^2$, then $\sqrt{m} = n$. We read $\sqrt{m}$ as 'the square root of m.'

square root of a number

- If $n^2 = m$, then n is a square root of m

KEY CONCEPTS

9.1 Simplify and Use Square Roots

- Note that the square root of a negative number is not a real number.
- Every positive number has two square roots, one positive and one negative. The positive square root of a positive number is the principal square root.
- We can estimate square roots using nearby perfect squares.
- We can approximate square roots using a calculator.
- When we use the radical sign to take the square root of a variable expression, we should specify that $x \geq 0$ to make sure we get the principal square root.

9.2 Simplify Square Roots

- **Simplified Square Root** $\sqrt{a}$ is considered simplified if a has no perfect-square factors.
- **Product Property of Square Roots** If *a*, *b* are non-negative real numbers, then
$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$
- **Simplify a Square Root Using the Product Property** To simplify a square root using the Product Property:
  Step 1. Find the largest perfect square factor of the radicand. Rewrite the radicand as a product using the perfect square factor.
  Step 2. Use the product rule to rewrite the radical as the product of two radicals.
  Step 3. Simplify the square root of the perfect square.
- **Quotient Property of Square Roots** If *a*, *b* are non-negative real numbers and $b \neq 0$, then
$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

- **Simplify a Square Root Using the Quotient Property** To simplify a square root using the Quotient Property:
  Step 1. Simplify the fraction in the radicand, if possible.
  Step 2. Use the Quotient Rule to rewrite the radical as the quotient of two radicals.
  Step 3. Simplify the radicals in the numerator and the denominator.

9.3 Add and Subtract Square Roots

- To add or subtract like square roots, add or subtract the coefficients and keep the like square root.
- Sometimes when we have to add or subtract square roots that do not appear to have like radicals, we find like radicals after simplifying the square roots.

9.4 Multiply Square Roots

- **Product Property of Square Roots** If *a*, *b* are nonnegative real numbers, then
$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad \text{and} \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$
- **Special formulas** for multiplying binomials and conjugates:
$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (a-b)(a+b) = a^2 - b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
- The FOIL method can be used to multiply binomials containing radicals.

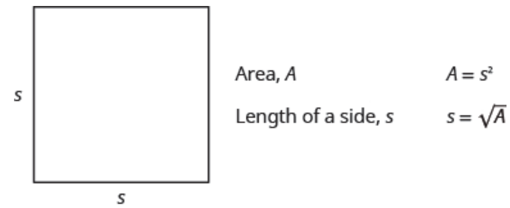
9.5 Divide Square Roots

- **Quotient Property of Square Roots**
  - If *a*, *b* are non-negative real numbers and $b \neq 0$, then
$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad \text{and} \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$
- **Simplified Square Roots**
  A square root is considered simplified if there are
  - no perfect square factors in the radicand
  - no fractions in the radicand
  - no square roots in the denominator of a fraction

9.6 Solve Equations with Square Roots

- **To Solve a Radical Equation:**
  Step 1. Isolate the radical on one side of the equation.
  Step 2. Square both sides of the equation.
  Step 3. Solve the new equation.
  Step 4. Check the answer. Some solutions obtained may not work in the original equation.
- **Solving Applications with Formulas**
  Step 1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  Step 2. **Identify** what we are looking for.
  Step 3. **Name** what we are looking for by choosing a variable to represent it.
  Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  Step 5. **Solve the equation** using good algebra techniques.
  Step 6. **Check** the answer in the problem and make sure it makes sense.
  Step 7. **Answer** the question with a complete sentence.

- **Area of a Square**



- **Falling Objects**
  - On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
  - If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

9.7 Higher Roots

- **Properties of**
- $\sqrt[n]{a}$ when n is an even number and
  - $a \geq 0$, then $\sqrt[n]{a}$ is a real number
  - $a < 0$, then $\sqrt[n]{a}$ is not a real number
  - When n is an odd number, $\sqrt[n]{a}$ is a real number for all values of *a*.
  - For any integer $n \geq 2$, when *n* is odd $\sqrt[n]{a^n} = a$
  - For any integer $n \geq 2$, when *n* is even $\sqrt[n]{a^n} = |a|$
- $\sqrt[n]{a}$ is considered simplified if *a* has no factors of m^n.
- **Product Property of *n*th Roots**

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \text{ and } \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

- **Quotient Property of *n*th Roots**

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \text{ and } \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

- To combine like radicals, simply add or subtract the coefficients while keeping the radical the same.

9.8 Rational Exponents

- **Summary of Exponent Properties**
- If a, b are real numbers and m, n are rational numbers, then
  - **Product Property** $a^m \cdot a^n = a^{m+n}$
  - **Power Property** $(a^m)^n = a^{m \cdot n}$
  - **Product to a Power** $(ab)^m = a^m b^m$
  - **Quotient Property**:

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0, \quad m > n$$

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad a \neq 0, \quad n > m$$

  - **Zero Exponent Definition** $a^0 = 1$, $a \neq 0$

- **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \quad b \neq 0$

REVIEW EXERCISES

9.1 Section 9.1 Simplify and Use Square Roots

Simplify Expressions with Square Roots

In the following exercises, simplify.

606. $\sqrt{64}$

607. $\sqrt{144}$

608. $-\sqrt{25}$

609. $-\sqrt{81}$

610. $\sqrt{-9}$

611. $\sqrt{-36}$

612. $\sqrt{64} + \sqrt{225}$

613. $\sqrt{64 + 225}$

Estimate Square Roots

In the following exercises, estimate each square root between two consecutive whole numbers.

614. $\sqrt{28}$

615. $\sqrt{155}$

Approximate Square Roots

In the following exercises, approximate each square root and round to two decimal places.

616. $\sqrt{15}$

617. $\sqrt{57}$

Simplify Variable Expressions with Square Roots

In the following exercises, simplify.

618. $\sqrt{q^2}$

619. $\sqrt{64b^2}$

620. $-\sqrt{121a^2}$

621. $\sqrt{225m^2n^2}$

622. $-\sqrt{100q^2}$

623. $\sqrt{49y^2}$

624. $\sqrt{4a^2b^2}$

625. $\sqrt{121c^2d^2}$

9.2 Section 9.2 Simplify Square Roots

Use the Product Property to Simplify Square Roots

In the following exercises, simplify.

626. $\sqrt{300}$

627. $\sqrt{98}$

628. $\sqrt{x^{13}}$

629. $\sqrt{y^{19}}$

630. $\sqrt{16m^4}$

631. $\sqrt{36n^{13}}$

632. $\sqrt{288m^{21}}$

633. $\sqrt{150n^7}$

634. $\sqrt{48r^5s^4}$

635. $\sqrt{108r^5s^3}$

636. $\dfrac{10 - \sqrt{50}}{5}$

637. $\dfrac{6 + \sqrt{72}}{6}$

Use the Quotient Property to Simplify Square Roots

In the following exercises, simplify.

638. $\sqrt{\dfrac{16}{25}}$

639. $\sqrt{\dfrac{81}{36}}$

640. $\sqrt{\dfrac{x^8}{x^4}}$

641. $\sqrt{\frac{y^6}{y^2}}$

642. $\sqrt{\frac{98p^6}{2p^2}}$

643. $\sqrt{\frac{72q^8}{2q^4}}$

644. $\sqrt{\frac{65}{121}}$

645. $\sqrt{\frac{26}{169}}$

646. $\sqrt{\frac{64x^4}{25x^2}}$

647. $\sqrt{\frac{36r^{10}}{16r^5}}$

648. $\sqrt{\frac{48p^3q^5}{27pq}}$

649. $\sqrt{\frac{12r^5s^7}{75r^2s}}$

9.3 Section 9.3 Add and Subtract Square Roots

Add and Subtract Like Square Roots

In the following exercises, simplify.

650. $3\sqrt{2} + \sqrt{2}$

651. $5\sqrt{5} + 7\sqrt{5}$

652. $4\sqrt{y} + 4\sqrt{y}$

653. $6\sqrt{m} - 2\sqrt{m}$

654. $-3\sqrt{7} + 2\sqrt{7} - \sqrt{7}$

655. $8\sqrt{13} + 2\sqrt{3} + 3\sqrt{13}$

656. $3\sqrt{5xy} - \sqrt{5xy} + 3\sqrt{5xy}$

657. $2\sqrt{3rs} + \sqrt{3rs} - 5\sqrt{rs}$

Add and Subtract Square Roots that Need Simplification

In the following exercises, simplify.

658. $\sqrt{32} + 3\sqrt{2}$

659. $\sqrt{8} + 3\sqrt{2}$

660. $\sqrt{72} + \sqrt{50}$

661. $\sqrt{48} + \sqrt{75}$

662. $3\sqrt{32} + \sqrt{98}$

663. $\frac{1}{3}\sqrt{27} - \frac{1}{8}\sqrt{192}$

664. $\sqrt{50y^5} - \sqrt{72y^5}$

665. $6\sqrt{18n^4} - 3\sqrt{8n^4} + n^2\sqrt{50}$

9.4 Section 9.4 Multiply Square Roots

Multiply Square Roots

In the following exercises, simplify.

666. $\sqrt{2} \cdot \sqrt{20}$

667. $2\sqrt{2} \cdot 6\sqrt{14}$

668. $\sqrt{2m^2} \cdot \sqrt{20m^4}$

669. $\left(6\sqrt{2y}\right)\left(3\sqrt{50y^3}\right)$

670. $\left(6\sqrt{3v^4}\right)\left(5\sqrt{30v}\right)$

671. $\left(\sqrt{8}\right)^2$

672. $\left(-\sqrt{10}\right)^2$

673. $\left(2\sqrt{5}\right)\left(5\sqrt{5}\right)$

674. $\left(-3\sqrt{3}\right)\left(5\sqrt{18}\right)$

Use Polynomial Multiplication to Multiply Square Roots

In the following exercises, simplify.

675. $10(2 - \sqrt{7})$

676. $\sqrt{3}\left(4 + \sqrt{12}\right)$

677. $\left(5 + \sqrt{2}\right)\left(3 - \sqrt{2}\right)$

678. $(5 - 3\sqrt{7})(1 - 2\sqrt{7})$

679. $(1 - 3\sqrt{x})(5 + 2\sqrt{x})$

680. $(3 + 4\sqrt{y})(10 - \sqrt{y})$

681. $(1 + 6\sqrt{p})^2$

682. $\left(2 - 6\sqrt{5}\right)^2$

683. $\left(3 + 2\sqrt{7}\right)\left(3 - 2\sqrt{7}\right)$

684. $\left(6 - \sqrt{11}\right)\left(6 + \sqrt{11}\right)$

9.5 Section 9.5 Divide Square Roots

Divide Square Roots

In the following exercises, simplify.

685. $\dfrac{\sqrt{75}}{10}$

686. $\dfrac{2-\sqrt{12}}{6}$

687. $\dfrac{\sqrt{48}}{\sqrt{27}}$

688. $\dfrac{\sqrt{75x^7}}{\sqrt{3x^3}}$

689. $\dfrac{\sqrt{20y^5}}{\sqrt{2y}}$

690. $\dfrac{\sqrt{98p^6q^4}}{\sqrt{2p^4q^8}}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

691. $\dfrac{10}{\sqrt{15}}$

692. $\dfrac{6}{\sqrt{6}}$

693. $\dfrac{5}{3\sqrt{5}}$

694. $\dfrac{10}{2\sqrt{6}}$

695. $\sqrt{\dfrac{3}{28}}$

696. $\sqrt{\dfrac{9}{75}}$

Rationalize a Two Term Denominator

In the following exercises, rationalize the denominator.

697. $\dfrac{4}{4+\sqrt{27}}$

698. $\dfrac{5}{2-\sqrt{10}}$

699. $\dfrac{4}{2-\sqrt{5}}$

700. $\dfrac{5}{4-\sqrt{8}}$

701. $\dfrac{\sqrt{2}}{\sqrt{p}+\sqrt{3}}$

702. $\dfrac{\sqrt{x}-\sqrt{2}}{\sqrt{x}+\sqrt{2}}$

9.6 Section 9.6 Solve Equations with Square Roots

Solve Radical Equations

In the following exercises, solve the equation.

703. $\sqrt{7z+1}=6$

704. $\sqrt{4u-2}-4=0$

705. $\sqrt{6m+4}-5=0$

706. $\sqrt{2u-3}+2=0$

707. $\sqrt{u-4}+4=u$

708. $\sqrt{v-9}+9=0$

709. $\sqrt{r-4}-r=-10$

710. $\sqrt{s-9}-s=-9$

711. $2\sqrt{2x-7}-4=8$

712. $\sqrt{2-x}=\sqrt{2x-7}$

713. $\sqrt{a}+3=\sqrt{a+9}$

714. $\sqrt{r}+3=\sqrt{r+4}$

715. $\sqrt{u}+2=\sqrt{u+5}$

716. $\sqrt{n+11}-1=\sqrt{n+4}$

717. $\sqrt{y+5}+1=\sqrt{2y+3}$

Use Square Roots in Applications

In the following exercises, solve. Round approximations to one decimal place.

718. A pallet of sod will cover an area of about 600 square feet. Trinh wants to order a pallet of sod to make a square lawn in his backyard. Use the formula $s=\sqrt{A}$ to find the length of each side of his lawn.

719. A helicopter dropped a package from a height of 900 feet above a stranded hiker. Use the formula $t=\dfrac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the hiker.

720. Officer Morales measured the skid marks of one of the cars involved in an accident. The length of the skid marks was 245 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied.

9.7 Section 9.7 Higher Roots

Simplify Expressions with Higher Roots

In the following exercises, simplify.

721.
ⓐ $\sqrt[6]{64}$
ⓑ $\sqrt[3]{64}$

722.
ⓐ $\sqrt[3]{-27}$
ⓑ $\sqrt[4]{-64}$

723.
ⓐ $\sqrt[9]{d^9}$
ⓑ $\sqrt[8]{v^8}$

724.
ⓐ $\sqrt[5]{a^{10}}$
ⓑ $\sqrt[3]{b^{27}}$

725.
ⓐ $\sqrt[4]{16x^8}$
ⓑ $\sqrt[6]{64y^{12}}$

726.
ⓐ $\sqrt[7]{128r^{14}}$
ⓑ $\sqrt[4]{81s^{24}}$

Use the Product Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.

727.
ⓐ $\sqrt[9]{d^9}$
ⓑ $\sqrt[11]{m^{17}}$

728.
ⓐ $\sqrt[3]{54}$
ⓑ $\sqrt[4]{128}$

729.
ⓐ $\sqrt[5]{64c^8}$
ⓑ $\sqrt[4]{48d^7}$

730.
ⓐ $\sqrt[3]{343q^7}$
ⓑ $\sqrt[6]{192r^9}$

731.
ⓐ $\sqrt[3]{-500}$
ⓑ $\sqrt[4]{-16}$

Use the Quotient Property to Simplify Expressions with Higher Roots

In the following exercises, simplify.

732. $\sqrt[5]{\dfrac{r^{10}}{r^5}}$

733. $\sqrt[3]{\dfrac{w^{12}}{w^2}}$

734. $\sqrt[4]{\dfrac{64y^8}{4y^5}}$

735. $\sqrt[3]{\dfrac{54z^9}{2z^3}}$

736. $\sqrt[6]{\dfrac{64a^7}{b^2}}$

Add and Subtract Higher Roots

In the following exercises, simplify.

737. $4\sqrt[5]{20} - 2\sqrt[5]{20}$

738. $4\sqrt[3]{18} + 3\sqrt[3]{18}$

739. $\sqrt[4]{1250} - \sqrt[4]{162}$

740. $\sqrt[3]{640c^5} - \sqrt[3]{-80c^3}$

741. $\sqrt[5]{96t^8} + \sqrt[5]{486t^4}$

9.8 Section 9.8 Rational Exponents

Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

742. $r^{\frac{1}{8}}$

743. $s^{\frac{1}{10}}$

In the following exercises, write with a rational exponent.

744. $\sqrt[5]{u}$

745. $\sqrt[6]{v}$

746. $\sqrt[3]{9m}$

747. $\sqrt[6]{10z}$

In the following exercises, simplify.

748. $16^{\frac{1}{4}}$

749. $32^{\frac{1}{5}}$

750. $(-125)^{\frac{1}{3}}$

751. $(125)^{-\frac{1}{3}}$

752. $(-9)^{\frac{1}{2}}$

753. $(36)^{-\frac{1}{2}}$

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

754. $\sqrt[3]{q^5}$

755. $\sqrt[5]{n^8}$

In the following exercises, simplify.

756. $27^{-\frac{2}{3}}$

757. $64^{\frac{5}{2}}$

758. $36^{\frac{3}{2}}$

759. $81^{-\frac{5}{2}}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

760. $3^{\frac{4}{5}} \cdot 3^{\frac{6}{5}}$

761. $\left(x^6\right)^{\frac{4}{3}}$

762. $\dfrac{z^{\frac{5}{2}}}{z^{\frac{7}{5}}}$

763. $\left(16s^{\frac{9}{4}}\right)^{\frac{1}{4}}$

764. $\left(m^8 n^{12}\right)^{\frac{1}{4}}$

765. $\dfrac{z^{\frac{2}{3}} \cdot z^{-\frac{1}{3}}}{z^{-\frac{5}{3}}}$

PRACTICE TEST

In the following exercises, simplify.

766. $\sqrt{81+144}$

767. $\sqrt{169m^4n^2}$

768. $\sqrt{36n^{13}}$

769. $3\sqrt{13}+5\sqrt{2}+\sqrt{13}$

770. $5\sqrt{20}+2\sqrt{125}$

771. $\left(3\sqrt{6y}\right)\left(2\sqrt{50y^3}\right)$

772. $(2-5\sqrt{x})(3+\sqrt{x})$

773. $(1-2\sqrt{q})^2$

774.
ⓐ $\sqrt[4]{a^{12}}$
ⓑ $\sqrt[3]{b^{21}}$

775.
ⓐ $\sqrt[4]{81x^{12}}$
ⓑ $\sqrt[6]{64y^{18}}$

776. $\sqrt{\dfrac{64r^{12}}{25r^6}}$

777. $\sqrt{\dfrac{14y^3}{7y}}$

778. $\dfrac{\sqrt[5]{256x^7}}{\sqrt[5]{4x^2}}$

779. $\sqrt[4]{512}-2\sqrt[4]{32}$

780.
ⓐ $256^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$

781. $49^{\frac{3}{2}}$

782. $25^{-\frac{5}{2}}$

783. $\dfrac{w^{\frac{3}{4}}}{w^{\frac{7}{4}}}$

784. $\left(27s^{\frac{3}{5}}\right)^{\frac{1}{3}}$

In the following exercises, rationalize the denominator.

785. $\dfrac{3}{2\sqrt{6}}$

786. $\dfrac{\sqrt{3}}{\sqrt{x}+\sqrt{5}}$

In the following exercises, solve.

787. $3\sqrt{2x-3}-20=7$

788. $\sqrt{3u-2}=\sqrt{5u+1}$

In the following exercise, solve.

789. A helicopter flying at an altitude of 600 feet dropped a package to a lifeboat. Use the formula $t=\dfrac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the hiker. Round your answer to the nearest tenth of a second.

10 QUADRATIC EQUATIONS

Figure 10.1 Fireworks accompany festive celebrations around the world. (Credit: modification of work by tlc, Flickr)

Chapter Outline

Introduction

The trajectories of fireworks are modeled by quadratic equations. The equations can be used to predict the maximum height of a firework and the number of seconds it will take from launch to explosion. In this chapter, we will study the properties of quadratic equations, solve them, graph them, and see how they are applied as models of various situations.

10.1 Solve Quadratic Equations Using the Square Root Property

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\sqrt{75}$.
   If you missed this problem, review **Example 9.12**.
2. Simplify: $\sqrt{\frac{64}{3}}$.
   If you missed this problem, review **Example 9.67**.
3. Factor: $4x^2 - 12x + 9$.
   If you missed this problem, review **Example 7.43**.

Quadratic equations are equations of the form $ax^2 + bx + c = 0$, where $a \neq 0$. They differ from linear equations by including a term with the variable raised to the second power. We use different methods to solve **quadratic equations** than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will use three other methods to solve quadratic equations.

Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

	$x^2 = 9$
Put the equation in standard form.	$x^2 - 9 = 0$
Factor the left side.	$(x-3)(x+3) = 0$
Use the Zero Product Property.	$(x-3) = 0, \quad (x+3) = 0$
Solve each equation.	$x = 3, \quad x = -3$
Combine the two solutions into $\pm$ form.	$x = \pm 3$
(The solution is read 'x is equal to positive or negative 3.')	

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

These equations are all of the form $x^2 = k$.
We defined the square root of a number in this way:

$$\text{If } n^2 = m, \text{ then } n \text{ is a square root of } m.$$

This leads to the **Square Root Property**.

Square Root Property

If $x^2 = k$, and $k \geq 0$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$: the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$.

Now, we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

	$x^2 = 9$
Use the Square Root Property.	$x = \pm\sqrt{9}$
Simplify the radical.	$x = \pm 3$
Rewrite to show the two solutions.	$x = 3, \quad x = -3$

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

	$x^2 = 7$
Use the Square Root Property.	$x = \pm\sqrt{7}$
Rewrite to show two solutions.	$x = \sqrt{7}, \quad x = -\sqrt{7}$
We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.	

EXAMPLE 10.1

Solve: $x^2 = 169$.

Solution

	$x^2 = 169$
Use the Square Root Property.	$x = \pm\sqrt{169}$
Simplify the radical.	$x = \pm 13$
Rewrite to show two solutions.	$x = 13, \quad x = -13$

TRY IT : : 10.1 Solve: $x^2 = 81$.

TRY IT : : 10.2 Solve: $y^2 = 121$.

EXAMPLE 10.2 HOW TO SOLVE A QUADRATIC EQUATION OF THE FORM $ax^2 = k$ USING THE SQUARE ROOT PROPERTY

Solve: $x^2 - 48 = 0$.

Solution

Step 1. Isolate the quadratic term and make its coefficient one.	Add 48 to both sides to get x^2 by itself.	$x^2 - 48 = 0$ $x^2 = 48$
Step 2. Use the Square Root Property.	Remember to add the ± symbol.	$x = \pm\sqrt{48}$
Step 3. Simplify the radical.		$x = \pm\sqrt{16} \cdot \sqrt{3}$ $x = \pm 4\sqrt{3}$ $x = 4\sqrt{3}, x = -4\sqrt{3}$
Step 4. Check the solutions.	Substitute in $x = 4\sqrt{3}$ and $x = -4\sqrt{3}$.	$x^2 - 48 = 0$ $\left(4\sqrt{3}\right)^2 - 48 \stackrel{?}{=} 0$ $16 \cdot 3 - 48 \stackrel{?}{=} 0$ $0 = 0$ ✓ $x^2 - 48 = 0$ $\left(-4\sqrt{3}\right)^2 - 48 \stackrel{?}{=} 0$ $16 \cdot 3 - 48 \stackrel{?}{=} 0$ $0 = 0$ ✓

TRY IT : : 10.3 Solve: $x^2 - 50 = 0$.

TRY IT : : 10.4 Solve: $y^2 - 27 = 0$.

HOW TO : : SOLVE A QUADRATIC EQUATION USING THE SQUARE ROOT PROPERTY.

Step 1. Isolate the quadratic term and make its coefficient one.
Step 2. Use Square Root Property.
Step 3. Simplify the radical.
Step 4. Check the solutions.

To use the Square Root Property, the coefficient of the variable term must equal 1. In the next example, we must divide both sides of the equation by 5 before using the Square Root Property.

EXAMPLE 10.3

Solve: $5m^2 = 80$.

Solution

The quadratic term is isolated.	$5m^2 = 80$
Divide by 5 to make its cofficient 1.	$\frac{5m^2}{5} = \frac{80}{5}$
Simplify.	$m^2 = 16$
Use the Square Root Property.	$m = \pm\sqrt{16}$
Simplify the radical.	$m = \pm 4$
Rewrite to show two solutions.	$m = 4, m = -4$
Check the solutions.	

$$5m^2 = 80 \qquad 5m^2 = 80$$
$$5(4)^2 \stackrel{?}{=} 80 \qquad 5(-4)^2 \stackrel{?}{=} 80$$
$$5 \cdot 16 \stackrel{?}{=} 80 \qquad 5 \cdot 16 \stackrel{?}{=} 80$$
$$80 = 80 \checkmark \qquad 80 = 80 \checkmark$$

TRY IT : : 10.5 Solve: $2x^2 = 98$.

TRY IT : : 10.6 Solve: $3z^2 = 108$.

The Square Root Property started by stating, 'If $x^2 = k$, and $k \geq 0$'. What will happen if $k < 0$? This will be the case in the next example.

EXAMPLE 10.4

Solve: $q^2 + 24 = 0$.

Solution

	$q^2 + 24 = 0$
Isolate the quadratic term.	$q^2 = -24$
Use the Square Root Property.	$q = \pm\sqrt{-24}$
The $\sqrt{-24}$ is not a real number.	There is no real solution.

TRY IT : : 10.7 Solve: $c^2 + 12 = 0$.

TRY IT : : 10.8 Solve: $d^2 + 81 = 0$.

Remember, we first isolate the quadratic term and then make the coefficient equal to one.

EXAMPLE 10.5

Solve: $\frac{2}{3}u^2 + 5 = 17$.

Solution

	$\frac{2}{3}u^2 + 5 = 17$
Isolate the quadratic term.	$\frac{2}{3}u^2 = 12$
Multiply by $\frac{3}{2}$ to make the coefficient 1.	$\frac{3}{2} \cdot \frac{2}{3}u^2 = \frac{3}{2} \cdot 12$
Simplify.	$u^2 = 18$
Use the Square Root Property.	$u = \pm\sqrt{18}$
Simplify the radical.	$u = \pm\sqrt{9}\sqrt{2}$
Simplify.	$u = \pm 3\sqrt{2}$
Rewrite to show two solutions.	$u = 3\sqrt{2},\ u = -3\sqrt{2}$
Check. $\frac{2}{3}u^2 + 5 = 17$; $\frac{2}{3}(3\sqrt{2})^2 + 5 \stackrel{?}{=} 17$; $\frac{2}{3} \cdot 18 + 5 \stackrel{?}{=} 17$; $12 + 5 \stackrel{?}{=} 17$; $17 = 17$ ✓ — $\frac{2}{3}u^2 + 5 = 17$; $\frac{2}{3}(-3\sqrt{2})^2 + 5 \stackrel{?}{=} 17$; $\frac{2}{3} \cdot 18 + 5 \stackrel{?}{=} 17$; $12 + 5 \stackrel{?}{=} 17$; $17 = 17$ ✓	

TRY IT : : 10.9 Solve: $\frac{1}{2}x^2 + 4 = 24$.

TRY IT : : 10.10 Solve: $\frac{3}{4}y^2 - 3 = 18$.

The solutions to some equations may have fractions inside the radicals. When this happens, we must rationalize the denominator.

EXAMPLE 10.6

Solve: $2c^2 - 4 = 45$.

Solution

	$2c^2 - 4 = 45$
Isolate the quadratic term.	$2c^2 = 49$
Divide by 2 to make the coefficient	$\dfrac{2c^2}{2} = \dfrac{49}{2}$
Simplify.	$c^2 = \dfrac{49}{2}$
Use the Square Root Property.	$c = \pm\sqrt{\dfrac{49}{2}}$
Simplify the radical.	$c = \pm\dfrac{\sqrt{49}}{\sqrt{2}}$
Rationalize the denominator.	$c = \pm\dfrac{\sqrt{49}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$
Simplify.	$c = \pm\dfrac{7\sqrt{2}}{2}$
Rewrite to show two solutions.	$c = \dfrac{7\sqrt{2}}{2},\quad c = -\dfrac{7\sqrt{2}}{2}$
Check. We leave the check for you.	

TRY IT : : 10.11 Solve: $5r^2 - 2 = 34$.

TRY IT : : 10.12 Solve: $3t^2 + 6 = 70$.

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

We can use the Square Root Property to solve an equation like $(x - 3)^2 = 16$, too. We will treat the whole binomial, $(x - 3)$, as the quadratic term.

EXAMPLE 10.7

Solve: $(x - 3)^2 = 16$.

Solution

	$(x-3)^2 = 16$
Use the Square Root Property.	$x - 3 = \pm\sqrt{16}$
Simplify.	$x - 3 = \pm 4$
Write as two equations.	$x - 3 = 4,\ x - 3 = -4$
Solve.	$x = 7,\ x = -1$
Check. $(7-3)^2 = 16$; $(4)^2 = 16$; $16 = 16$ ✓ $(-1-3)^2 = 16$; $(-4)^2 = 16$; $16 = 16$ ✓	

TRY IT : : 10.13 Solve: $(q + 5)^2 = 1$.

TRY IT : : 10.14 Solve: $(r-3)^2 = 25$.

EXAMPLE 10.8

Solve: $(y-7)^2 = 12$.

Solution

	$(y-7)^2 = 12$
Use the Square Root Property.	$y-7 = \pm\sqrt{12}$
Simplify the radical.	$y-7 = \pm 2\sqrt{3}$
Solve for *y*.	$y = 7 \pm 2\sqrt{3}$
Rewrite to show two solutions.	$y = 7+2\sqrt{3},\ y = 7-2\sqrt{3}$
Check. $(y-7)^2 = 12$; $(7+2\sqrt{3}-7)^2 \overset{?}{=} 12$; $(2\sqrt{3})^2 \overset{?}{=} 12$; $12 = 12$ ✓ — $(y-7)^2 = 12$; $(7-2\sqrt{3}-7)^2 \overset{?}{=} 12$; $(-2\sqrt{3})^2 \overset{?}{=} 12$; $12 = 12$ ✓	

TRY IT : : 10.15 Solve: $(a-3)^2 = 18$.

TRY IT : : 10.16 Solve: $(b+2)^2 = 40$.

Remember, when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

EXAMPLE 10.9

Solve: $\left(x-\frac{1}{2}\right)^2 = \frac{5}{4}$.

Solution

	$\left(x-\frac{1}{2}\right)^2 = \frac{5}{4}$
Use the Square Root Property.	$x-\frac{1}{2} = \pm\sqrt{\frac{5}{4}}$
Rewrite the radical as a fraction of square roots.	$x-\frac{1}{2} = \pm\frac{\sqrt{5}}{\sqrt{4}}$
Simplify the radical.	$x-\frac{1}{2} = \pm\frac{\sqrt{5}}{2}$
Solve for x.	$x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$
Rewrite to show two solutions.	$x = \frac{1}{2}+\frac{\sqrt{5}}{2},\quad x = \frac{1}{2}-\frac{\sqrt{5}}{2}$
Check. We leave the check for you.	

TRY IT : : 10.17 Solve: $\left(x-\frac{1}{3}\right)^2=\frac{5}{9}$.

TRY IT : : 10.18 Solve: $\left(y-\frac{3}{4}\right)^2=\frac{7}{16}$.

We will start the solution to the next example by isolating the binomial.

EXAMPLE 10.10

Solve: $(x-2)^2+3=30$.

Solution

	$(x-2)^2+3=30$
Isolate the binomial term.	$(x-2)^2=27$
Use the Square Root Property.	$x-2=\pm\sqrt{27}$
Simplify the radical.	$x-2=\pm 3\sqrt{3}$
Solve for x.	$x=2\pm 3\sqrt{3}$
Rewrite to show two solutions.	$x=2+3\sqrt{3},\quad x=2-3\sqrt{3}$
Check. We leave the check for you.	

TRY IT : : 10.19 Solve: $(a-5)^2+4=24$.

TRY IT : : 10.20 Solve: $(b-3)^2-8=24$.

EXAMPLE 10.11

Solve: $(3v-7)^2=-12$.

Solution

Use the Square Root Property.	$(3v-7)^2=-12$
	$3v-7=\pm\sqrt{-12}$
The $\sqrt{-12}$ is not a real number.	There is no real solution.

TRY IT : : 10.21 Solve: $(3r+4)^2=-8$.

TRY IT : : 10.22 Solve: $(2t-8)^2=-10$.

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

EXAMPLE 10.12

Solve: $p^2-10p+25=18$.

Solution

The left side of the equation is a perfect square trinomial. We will factor it first.

	$p^2 - 10p + 25 = 18$
Factor the perfect square trinomial.	$(p-5)^2 = 18$
Use the Square Root Property.	$p - 5 = \pm\sqrt{18}$
Simplify the radical.	$p - 5 = \pm 3\sqrt{2}$
Solve for p.	$p = 5 \pm 3\sqrt{2}$
Rewrite to show two solutions.	$p = 5 + 3\sqrt{2},\quad p = 5 - 3\sqrt{2}$
Check. We leave the check for you.	

TRY IT : : 10.23 Solve: $x^2 - 6x + 9 = 12$.

TRY IT : : 10.24 Solve: $y^2 + 12y + 36 = 32$.

EXAMPLE 10.13

Solve: $4n^2 + 4n + 1 = 16$.

Solution

Again, we notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2 + 4n + 1 = 16$
Factor the perfect square trinomial.	$(2n+1)^2 = 16$
Use the Square Root Property.	$2n + 1 = \pm\sqrt{16}$
Simplify the radical.	$2n + 1 = \pm 4$
Solve for n.	$2n = -1 \pm 4$
Divide each side by 2.	$\frac{2n}{2} = \frac{-1 \pm 4}{2}$ $n = \frac{-1 \pm 4}{2}$
Rewrite to show two solutions.	$n = \frac{-1+4}{2},\ n = \frac{-1-4}{2}$
Simplify each equation.	$n = \frac{3}{2},\ n = -\frac{5}{2}$
Check.	

$$\begin{array}{ll}
4n^2 + 4n + 1 = 16 & 4n^2 + 4n + 1 = 16 \\
4\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16 & 4\left(-\frac{5}{2}\right)^2 + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16 \\
4\left(\frac{9}{4}\right) + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16 & 4\left(\frac{25}{4}\right) + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16 \\
9 + 6 + 1 \stackrel{?}{=} 16 & 25 - 10 + 1 \stackrel{?}{=} 16 \\
16 = 16\checkmark & 16 = 16\checkmark
\end{array}$$

TRY IT : : 10.25 Solve: $9m^2 - 12m + 4 = 25$.

TRY IT : : 10.26 Solve: $16n^2 + 40n + 25 = 4$.

MEDIA : :

Access these online resources for additional instruction and practice with solving quadratic equations:

- **Solving Quadratic Equations: Solving by Taking Square Roots (https://openstax.org/l/25Solvebysqroot)**
- **Using Square Roots to Solve Quadratic Equations (https://openstax.org/l/25Usesqroots)**
- **Solving Quadratic Equations: The Square Root Method (https://openstax.org/l/25Sqrtproperty)**

10.1 EXERCISES

Practice Makes Perfect

Solve Quadratic Equations of the form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve the following quadratic equations.

1. $a^2 = 49$

2. $b^2 = 144$

3. $r^2 - 24 = 0$

4. $t^2 - 75 = 0$

5. $u^2 - 300 = 0$

6. $v^2 - 80 = 0$

7. $4m^2 = 36$

8. $3n^2 = 48$

9. $x^2 + 20 = 0$

10. $y^2 + 64 = 0$

11. $\frac{2}{5}a^2 + 3 = 11$

12. $\frac{3}{2}b^2 - 7 = 41$

13. $7p^2 + 10 = 26$

14. $2q^2 + 5 = 30$

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

In the following exercises, solve the following quadratic equations.

15. $(x + 2)^2 = 9$

16. $(y - 5)^2 = 36$

17. $(u - 6)^2 = 64$

18. $(v + 10)^2 = 121$

19. $(m - 6)^2 = 20$

20. $(n + 5)^2 = 32$

21. $\left(r - \frac{1}{2}\right)^2 = \frac{3}{4}$

22. $\left(t - \frac{5}{6}\right)^2 = \frac{11}{25}$

23. $(a - 7)^2 + 5 = 55$

24. $(b - 1)^2 - 9 = 39$

25. $(5c + 1)^2 = -27$

26. $(8d - 6)^2 = -24$

27. $m^2 - 4m + 4 = 8$

28. $n^2 + 8n + 16 = 27$

29. $25x^2 - 30x + 9 = 36$

30. $9y^2 + 12y + 4 = 9$

Mixed Practice

In the following exercises, solve using the Square Root Property.

31. $2r^2 = 32$

32. $4t^2 = 16$

33. $(a - 4)^2 = 28$

34. $(b + 7)^2 = 8$

35. $9w^2 - 24w + 16 = 1$

36. $4z^2 + 4z + 1 = 49$

37. $a^2 - 18 = 0$

38. $b^2 - 108 = 0$

39. $\left(p - \frac{1}{3}\right)^2 = \frac{7}{9}$

40. $\left(q - \frac{3}{5}\right)^2 = \frac{3}{4}$

41. $m^2 + 12 = 0$

42. $n^2 + 48 = 0$

43. $u^2 - 14u + 49 = 72$

44. $v^2 + 18v + 81 = 50$

45. $(m - 4)^2 + 3 = 15$

46. $(n-7)^2-8=64$

47. $(x+5)^2=4$

48. $(y-4)^2=64$

49. $6c^2+4=29$

50. $2d^2-4=77$

51. $(x-6)^2+7=3$

52. $(y-4)^2+10=9$

Everyday Math

53. Paola has enough mulch to cover 48 square feet. She wants to use it to make three square vegetable gardens of equal sizes. Solve the equation $3s^2=48$ to find s, the length of each garden side.

54. Kathy is drawing up the blueprints for a house she is designing. She wants to have four square windows of equal size in the living room, with a total area of 64 square feet. Solve the equation $4s^2=64$ to find s, the length of the sides of the windows.

Writing Exercises

55. Explain why the equation $x^2+12=8$ has no solution.

56. Explain why the equation $y^2+8=12$ has two solutions.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations of the form $ax^2 = k$ using the square root property.			
solve quadratic equations of the form $a(x-h)^2 = k$ using the square root property.			

ⓑ *If most of your checks were:*

...confidently: *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

...with some help: *This must be addressed quickly because topics you do not master become potholes in your road to success. In math, every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no-I don't get it! *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

10.2 Solve Quadratic Equations by Completing the Square

Learning Objectives

By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square
- Solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square

Be Prepared!

Before you get started, take this readiness quiz. If you miss a problem, go back to the section listed and review the material.

1. Simplify $(x + 12)^2$.
   If you missed this problem, review **Example 6.47**.
2. Factor $y^2 - 18y + 81$.
   If you missed this problem, review **Example 7.42**.
3. Factor $5n^2 + 40n + 80$.
   If you missed this problem, review **Example 7.46**.

So far, we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called ‘completing the square.’

Complete The Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y - 7)^2 = 12$ because the left side was a perfect square.

$$\begin{aligned} (y-7)^2 &= 12 \\ y-7 &= \pm\sqrt{12} \\ y-7 &= \pm 2\sqrt{3} \\ y &= 7 \pm 2\sqrt{3} \end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x - k)^2$ in order to use the square root property.

$$\begin{aligned} x^2 - 10x + 25 &= 18 \\ (x-5)^2 &= 18 \end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let’s study the binomial square pattern we have used many times. We will look at two examples.

$$\begin{array}{ll} (x+9)^2 & (y-7)^2 \\ (x+9)(x+9) & (y-7)(y-7) \\ x^2+9x+9x+81 & y^2-7y-7y+49 \\ x^2+18x+81 & y^2-14y+49 \end{array}$$

Binomial Squares Pattern

If a, b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2 \times (\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$\underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2 \times (\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

We can use this pattern to “make” a perfect square.

We will start with the expression $x^2 + 6x$. Since there is a plus sign between the two terms, we will use the $(a+b)^2$ pattern.

$$a^2 + 2ab + b^2 = (a+b)^2$$

Notice that the first term of $x^2 + 6x$ is a square, x^2.

We now know $a = x$.

What number can we add to $x^2 + 6x$ to make a perfect square trinomial?

$$\overset{a^2 + 2ab + b^2}{x^2 + 6x + \_\_}$$

The middle term of the Binomial Squares Pattern, $2ab$, is twice the product of the two terms of the binomial. This means twice the product of x and some number is $6x$. So, two times some number must be six. The number we need is $\frac{1}{2} \cdot 6 = 3.$ The second term in the binomial, $b,$ must be 3.

$$\overset{a^2 + \quad 2ab \quad + \; b^2}{x^2 + 2 \cdot 3 \cdot x + \_\_}$$

We now know $b = 3$.

Now, we just square the second term of the binomial to get the last term of the perfect square trinomial, so we square three to get the last term, nine.

$$\overset{a^2 + 2ab + b^2}{x^2 + 6x + 9}$$

We can now factor to

$$\overset{(a+b)^2}{(x+3)^2}$$

So, we found that adding nine to $x^2 + 6x$ ‘completes the square,’ and we write it as $(x+3)^2$.

HOW TO : : COMPLETE A SQUARE.

To complete the square of $x^2 + bx$:

Step 1. Identify b, the coefficient of x.

Step 2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.

Step 3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$.

EXAMPLE 10.14

Complete the square to make a perfect square trinomial. Then, write the result as a binomial square.

$x^2 + 14x$

Solution

The coefficient of x is 14.	$\underset{x^2+14x}{x^2+\ bx}$
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2} \cdot 14\right)^2$ $(7)^2$ 49	
Add 49 to the binomial to complete the square.	$x^2 + 14x + 49$
Rewrite as a binomial square.	$(x + 7)^2$

TRY IT : : 10.27

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$y^2 + 12y$

TRY IT : : 10.28

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$z^2 + 8z$

EXAMPLE 10.15

Complete the square to make a perfect square trinomial. Then, write the result as a binomial squared. $m^2 - 26m$

Solution

The coefficient of m is −26.	$\underbrace{m^2 - 26m}_{x^2 - bx}$
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2} \cdot (-26)\right)^2$ $(-13)^2$ 169	
Add 169 to the binomial to complete the square.	$m^2 - 26m + 169$
Rewrite as a binomial square.	$(m - 13)^2$

TRY IT : : 10.29

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$a^2 - 20a$

TRY IT : : 10.30

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$b^2 - 4b$

EXAMPLE 10.16

Complete the square to make a perfect square trinomial. Then, write the result as a binomial squared.

$u^2 - 9u$

Solution

The coefficient of u is −9.	$\underbrace{u^2 - 9u}_{x^2 + bx}$
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2} \cdot (-9)\right)^2$ $\left(-\frac{9}{2}\right)^2$ $\frac{81}{4}$	
Add $\frac{81}{4}$ to the binomial to complete the square.	$u^2 - 9u + \frac{81}{4}$
Rewrite as a binomial square.	$\left(u - \frac{9}{2}\right)^2$

TRY IT : : 10.31

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$m^2 - 5m$

TRY IT : : 10.32

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$n^2 + 13n$

EXAMPLE 10.17

Complete the square to make a perfect square trinomial. Then, write the result as a binomial squared.

$p^2 + \frac{1}{2}p$

Solution

The coefficient of p is $\frac{1}{2}$.	$\overset{x^2}{p^2} \overset{+}{+} \overset{bx}{\frac{1}{2}p}$
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2} \cdot \frac{1}{2}\right)^2$ $\left(\frac{1}{4}\right)^2$ $\frac{1}{16}$	
Add $\frac{1}{16}$ to the binomial to complete the square.	$p^2 + \frac{1}{2}p + \frac{1}{16}$
Rewrite as a binomial square.	$\left(p + \frac{1}{4}\right)^2$

TRY IT : : 10.33

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$p^2 + \frac{1}{4}p$

TRY IT : : 10.34

Complete the square to make a perfect square trinomial. Write the result as a binomial square.

$q^2 - \frac{2}{3}q$

Solve Quadratic Equations of the Form $x^2 + bx + c = 0$ by completing the square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a quadratic equation by completing the square, too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2 + 6x = 40$ and we want to complete the square on the left, we will add nine to both sides of the equation.

$$x^2 + 6x = 40$$
$$x^2 + 6x + \_\_ = 40 + \_\_$$
$$x^2 + 6x + 9 = 40 + 9$$

Then, we factor on the left and simplify on the right.

$$(x + 3)^2 = 49$$

Now the equation is in the form to solve using the Square Root Property. Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

EXAMPLE 10.18 HOW TO SOLVE A QUADRATIC EQUATION OF THE FORM $x^2 + bx + c = 0$ BY COMPLETING THE SQUARE

Solve $x^2 + 8x = 48$ by completing the square.

Solution

Step 1. Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$\underbrace{x^2 + 8x}_{x^2+bx} = \underbrace{48}_{c}$
Step 2. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	Take half of 8 and square it. $4^2 = 16$ Add 16 to BOTH sides of the equation.	$x^2 + 8x + \underbrace{\_\_\_}_{\left(\frac{1}{2}\cdot 8\right)^2} = 48$ $x^2 + 8x + 16 = 48 + 16$
Step 3. Factor the perfect square trinomial as a binomial square.	$x^2 + 8x + 16 = (x + 4)^2$ Add the terms on the right.	$(x + 4)^2 = 64$
Step 4. Use the Square Root Property.		$x + 4 = \pm\sqrt{64}$
Step 5. Simplify the radical and then solve the two resulting equations.		$x + 4 = \pm 8$ $x + 4 = 8 \quad x + 4 = -8$ $x = 4 \quad x = -12$
Step 6. Check the solutions.	Put each answer in the original equation to check. Substitute $x = 4$.	$x^2 + 8x = 48$ $(4)^2 + 8(4) \stackrel{?}{=} 48$ $16 + 32 \stackrel{?}{=} 48$ $48 = 48$ ✓
	Substitute $x = -12$.	$x^2 + 8x = 48$ $(-12)^2 + 8(-12) \stackrel{?}{=} 48$ $144 - 96 \stackrel{?}{=} 48$ $48 = 48$ ✓

TRY IT : : 10.35 Solve $c^2 + 4c = 5$ by completing the square.

TRY IT : : 10.36 Solve $d^2 + 10d = -9$ by completing the square.

HOW TO : : SOLVE A QUADRATIC EQUATION OF THE FORM $x^2 + bx + c = 0$ BY COMPLETING THE SQUARE.

Step 1. Isolate the variable terms on one side and the constant terms on the other.

Step 2. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.

Step 3. Factor the perfect square trinomial as a binomial square.

Step 4. Use the Square Root Property.

Step 5. Simplify the radical and then solve the two resulting equations.

Step 6. Check the solutions.

EXAMPLE 10.19

Solve $y^2 - 6y = 16$ by completing the square.

Solution

The variable terms are on the left side.	$\overset{x^2 - bx}{y^2 - 6y} = \overset{c}{16}$
Take half of -6 and square it. $\left(\frac{1}{2}(-6)\right)^2 = 9$	$y^2 - 6y + \underset{\left(\frac{1}{2} \cdot (-6)\right)^2}{\underline{\qquad}} = 16$
Add 9 to both sides.	$y^2 - 6y + 9 = 16 + 9$
Factor the perfect square trinomial as a binomial square.	$(y - 3)^2 = 25$
Use the Square Root Property.	$y - 3 = \pm\sqrt{25}$
Simplify the radical.	$y - 3 = \pm 5$
Solve for *y*.	$y = 3 \pm 5$
Rewrite to show two solutions.	$y = 3 + 5, y = 3 - 5$
Solve the equations.	$y = 8, y = -2$

Check.

$$\begin{array}{rr} y^2 - 6y = 16 & y^2 - 6y = 16 \\ 8^2 - 6 \cdot 8 \stackrel{?}{=} 16 & (-2)^2 - 6(-2) \stackrel{?}{=} 16 \\ 64 - 48 \stackrel{?}{=} 16 & 4 + 12 \stackrel{?}{=} 16 \\ 16 = 16 \checkmark & 16 = 16 \checkmark \end{array}$$

TRY IT : : 10.37 Solve $r^2 - 4r = 12$ by completing the square.

TRY IT : : 10.38 Solve $t^2 - 10t = 11$ by completing the square.

EXAMPLE 10.20

Solve $x^2 + 4x = -21$ by completing the square.

Solution

The variable terms are on the left side.	$\underbrace{x^2+4x}_{x^2+bx}=\underbrace{-21}_{c}$
Take half of 4 and square it. $\left(\frac{1}{2}(4)\right)^2=4$	$x^2+4x+\underset{\left(\frac{1}{2}\cdot 4\right)^2}{\underline{\qquad}}=-21$
Add 4 to both sides.	$x^2+4x+4=-21+4$
Factor the perfect square trinomial as a binomial square.	$(x+2)^2=-17$
Use the Square Root Property.	$x+2=\pm\sqrt{-17}$
We cannot take the square root of a negative number.	There is no real solution.

TRY IT : : 10.39 Solve $y^2-10y=-35$ by completing the square.

TRY IT : : 10.40 Solve $z^2+8z=-19$ by completing the square.

In the previous example, there was no real solution because $(x+k)^2$ was equal to a negative number.

EXAMPLE 10.21

Solve $p^2-18p=-6$ by completing the square.

Solution

The variable terms are on the left side.	$\underbrace{P^2-18P}_{x^2+bx}=\underbrace{-6}_{c}$
Take half of -18 and square it. $\left(\frac{1}{2}(-18)\right)^2=81$	$P^2-18p+\underset{\left(\frac{1}{2}\cdot(-18)\right)^2}{\underline{\qquad}}=-6$
Add 81 to both sides.	$p^2-18p+81=-6+81$
Factor the perfect square trinomial as a binomial square.	$(p-9)^2=75$
Use the Square Root Property.	$p-9=\pm\sqrt{75}$
Simplify the radical.	$p-9=\pm5\sqrt{3}$
Solve for *p*.	$p=9\pm5\sqrt{3}$
Rewrite to show two solutions.	$p=9+5\sqrt{3},\ p=9-5\sqrt{3}$

Check.

$$\begin{array}{rr} p^2-18p=-6 & p^2-18p=-6 \\ \left(9+5\sqrt{3}\right)^2-18\left(9+5\sqrt{3}\right)\stackrel{?}{=}-6 & \left(9-5\sqrt{3}\right)^2-18\left(9-5\sqrt{3}\right)\stackrel{?}{=}-6 \\ 81+90\sqrt{3}+75-162-90\sqrt{3}\stackrel{?}{=}-6 & 81-90\sqrt{3}+75-162+90\sqrt{3}\stackrel{?}{=}-6 \\ -6=-6\checkmark & -6=-6\checkmark \end{array}$$

Another way to check this would be to use a calculator. Evaluate p^2-18p for both of the solutions. The answer should be -6.

TRY IT : : 10.41 Solve $x^2 - 16x = -16$ by completing the square.

TRY IT : : 10.42 Solve $y^2 + 8y = 11$ by completing the square.

We will start the next example by isolating the variable terms on the left side of the equation.

EXAMPLE 10.22

Solve $x^2 + 10x + 4 = 15$ by completing the square.

Solution

The variable terms are on the left side.	$x^2 + 10x + 4 = 15$
Subtract 4 to get the constant terms on the right side.	$x^2 + 10x \quad = 11$
Take half of 10 and square it. $\left(\frac{1}{2}(10)\right)^2 = 25$	$x^2 + 10x + \underline{\quad\quad} = 11$ $\left(\frac{1}{2} \cdot (10)\right)^2$
Add 25 to both sides.	$x^2 + 10x + 25 = 11 + 25$
Factor the perfect square trinomial as a binomial square.	$(x + 5)^2 = 36$
Use the Square Root Property.	$x + 5 = \pm\sqrt{36}$
Simplify the radical.	$x + 5 = \pm\sqrt{36}$
Solve for *x*.	$x = -5 \pm 6$
Rewrite to show two equations.	$x = -5 + 6,\ x = -5 - 6$
Solve the equations.	$x = 1,\ x = -11$

Check.

$$x^2 + 10x + 4 = 15 \qquad\qquad x^2 + 10x + 4 = 15$$
$$(1)^2 + 10(1) + 4 \stackrel{?}{=} 15 \qquad\qquad (-11)^2 + 10(-11) + 4 \stackrel{?}{=} 15$$
$$1 + 10 + 4 \stackrel{?}{=} 15 \qquad\qquad 121 - 110 + 4 \stackrel{?}{=} 15$$
$$15 = 15 \checkmark \qquad\qquad 15 = 15 \checkmark$$

TRY IT : : 10.43 Solve $a^2 + 4a + 9 = 30$ by completing the square.

TRY IT : : 10.44 Solve $b^2 + 8b - 4 = 16$ by completing the square.

To solve the next equation, we must first collect all the variable terms to the left side of the equation. Then, we proceed as we did in the previous examples.

EXAMPLE 10.23

Solve $n^2 = 3n + 11$ by completing the square.

Solution

	$n^2 = 3n + 11$
Subtract $3n$ to get the variable terms on the left side.	$n^2 - 3n \quad = 11$
Take half of -3 and square it. $\left(\frac{1}{2}(-3)\right)^2 = \frac{9}{4}$	$n^2 - 3n + \underline{\qquad} = 11$ $\left(\frac{1}{2}\cdot(-3)\right)^2$
Add $\frac{9}{4}$ to both sides.	$n^2 - 3n + \frac{9}{4} = 11 + \frac{9}{4}$
Factor the perfect square trinomial as a binomial square.	$\left(n - \frac{3}{2}\right)^2 = \frac{44}{4} + \frac{9}{4}$
Add the fractions on the right side.	$\left(n - \frac{3}{2}\right)^2 = \frac{53}{4}$
Use the Square Root Property.	$n - \frac{3}{2} = \pm\sqrt{\frac{53}{4}}$
Simplify the radical.	$n - \frac{3}{2} = \pm\frac{\sqrt{53}}{2}$
Solve for n.	$n = \frac{3}{2} + \frac{\sqrt{53}}{2}$
Rewrite to show two equations.	$n = \frac{3}{2} + \frac{\sqrt{53}}{2},\ n = \frac{3}{2} - \frac{\sqrt{53}}{2}$
Check. We leave the check for you!	

TRY IT : : 10.45 Solve $p^2 = 5p + 9$ by completing the square.

TRY IT : : 10.46 Solve $q^2 = 7q - 3$ by completing the square.

Notice that the left side of the next equation is in factored form. But the right side is not zero, so we cannot use the Zero Product Property. Instead, we multiply the factors and then put the equation into the standard form to solve by completing the square.

EXAMPLE 10.24

Solve $(x - 3)(x + 5) = 9$ by completing the square.

Solution

	$(x-3)(x+5)=9$
We multiply binomials on the left.	$x^2+2x-15=9$
Add 15 to get the variable terms on the left side.	$x^2+2x \qquad =24$
Take half of 2 and square it. $\left(\frac{1}{2}(2)\right)^2=1$	$x^2+2x+\underline{\qquad}=24$ $\left(\frac{1}{2}\cdot(2)\right)^2$
Add 1 to both sides.	$x^2+2x+1=24+1$
Factor the perfect square trinomial as a binomial square.	$(x+1)^2=25$
Use the Square Root Property.	$x+1=\pm\sqrt{25}$
Solve for x.	$x=-1\pm5$
Rewite to show two solutions.	$x=-1+5,\ x=-1-5$
Simplify.	$x=4,\ x=-6$
Check. We leave the check for you!	

TRY IT : : 10.47 Solve $(c-2)(c+8)=7$ by completing the square.

TRY IT : : 10.48 Solve $(d-7)(d+3)=56$ by completing the square.

Solve Quadratic Equations of the form $ax^2+bx+c=0$ by completing the square

The process of completing the square works best when the leading coefficient is one, so the left side of the equation is of the form x^2+bx+c. If the x^2 term has a coefficient, we take some preliminary steps to make the coefficient equal to one.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

EXAMPLE 10.25

Solve $3x^2-12x-15=0$ by completing the square.

Solution

To complete the square, we need the coefficient of x^2 to be one. If we factor out the coefficient of x^2 as a common factor, we can continue with solving the equation by completing the square.

	$3x^2 - 12x - 15 = 0$
Factor out the greatest common factor.	$3(x^2 - 4x - 5) = 0$
Divide both sides by 3 to isolate the trinomial.	$\frac{3(x^2 - 4x - 5)}{3} = \frac{0}{3}$
Simplify.	$x^2 - 4x - 5 = 0$
Subtract 5 to get the constant terms on the right.	$x^2 - 4x \quad = 5$
Take half of 4 and square it. $\left(\frac{1}{2}(4)\right)^2 = 4$	$x^2 - 4x + \underline{\quad\quad} = 5$ $\left(\frac{1}{2} \cdot (4)\right)^2$
Add 4 to both sides.	$x^2 - 4x + 4 = 5 + 4$
Factor the perfect square trinomial as a binomial square.	$(x - 2)^2 = 9$
Use the Square Root Property.	$x - 2 = \pm\sqrt{9}$
Solve for *x*.	$x - 2 = \pm 3$
Rewrite to show 2 solutions.	$x = 2 + 3, x = 2 - 3$
Simplify.	$x = 5, x = -1$
Check.	

$x = 5$	$x = -1$
$3x^2 - 12x - 15 = 0$	$3x^2 - 12x - 15 = 0$
$3(5)^2 - 12(5) - 15 \stackrel{?}{=} 0$	$3(-1)^2 - 12(-1) - 15 \stackrel{?}{=} 0$
$75 - 60 - 15 \stackrel{?}{=} 0$	$3 + 12 - 15 \stackrel{?}{=} 0$
$0 = 0$ ✓	$0 = 0$ ✓

TRY IT : : 10.49 Solve $2m^2 + 16m - 8 = 0$ by completing the square.

TRY IT : : 10.50 Solve $4n^2 - 24n - 56 = 8$ by completing the square.

To complete the square, the leading coefficient must be one. When the leading coefficient is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient. This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

EXAMPLE 10.26

Solve $2x^2 - 3x = 20$ by completing the square.

Solution

Again, our first step will be to make the coefficient of x^2 be one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$2x^2 - 3x = 20$
Divide both sides by 2 to get the coefficient of x^2 to be 1.	$\frac{2x^2 - 3x}{2} = \frac{20}{2}$
Simplify.	$x^2 - \frac{3}{2}x = 10$
Take half of $-\frac{3}{2}$ and square it. $\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$	$x^2 - \frac{3}{2}x + \underline{\qquad} = 10$ $\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2$
Add $\frac{9}{16}$ to both sides.	$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$
Factor the perfect square trinomial as a binomial square.	$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$
Add the fractions on the right side.	$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$
Use the Square Root Property.	$x - \frac{3}{4} = \pm\sqrt{\frac{169}{16}}$
Simplify the radical.	$x - \frac{3}{4} = \pm\frac{13}{4}$
Solve for *x*.	$x = \frac{3}{4} \pm \frac{13}{4}$
Rewrite to show 2 solutions.	$x = \frac{3}{4} + \frac{13}{4}, x = \frac{3}{4} - \frac{13}{4}$
Simplify.	$x = 4, x = -\frac{5}{2}$
Check. We leave the check for you.	

TRY IT : : 10.51 Solve $3r^2 - 2r = 21$ by completing the square.

TRY IT : : 10.52 Solve $4t^2 + 2t = 20$ by completing the square.

EXAMPLE 10.27

Solve $3x^2 + 2x = 4$ by completing the square.

Solution

Again, our first step will be to make the coefficient of x^2 be one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$3x^2+2x=4$
Divide both sides by 3 to make the coefficient of x^2 equal 1.	$\frac{3x^2+2x}{3}=\frac{4}{3}$
Simplify.	$x^2+\frac{2}{3}x=\frac{4}{3}$
Take half of $\frac{2}{3}$ and square it. $\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2=\frac{1}{9}$	$x^2+\frac{2}{3}x+\underline{\quad\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2\quad}=\frac{4}{3}$
Add $\frac{1}{9}$ to both sides.	$x^2+\frac{2}{3}x+\frac{1}{9}=\frac{4}{3}+\frac{1}{9}$
Factor the perfect square trinomial as a binomial square.	$\left(x+\frac{1}{3}\right)^2=\frac{12}{9}+\frac{1}{9}$
Use the Square Root Property.	$x+\frac{1}{3}=\pm\sqrt{\frac{13}{9}}$
Simplify the radical.	$x+\frac{1}{3}=\pm\frac{\sqrt{13}}{3}$
Solve for *x*.	$x=-\frac{1}{3}\pm\frac{\sqrt{13}}{3}$
Rewrite to show 2 solutions.	$x=-\frac{1}{3}+\frac{\sqrt{13}}{3},\ x=-\frac{1}{3}-\frac{\sqrt{13}}{3}$
Check. We leave the check for you.	

TRY IT : : 10.53 Solve $4x^2+3x=12$ by completing the square.

TRY IT : : 10.54 Solve $5y^2+3y=10$ by completing the square.

MEDIA : :

Access these online resources for additional instruction and practice with solving quadratic equations by completing the square:

- **Introduction to the method of completing the square (https://openstax.org/l/25Completethesq)**
- **How to Solve By Completing the Square (https://openstax.org/l/25Solvebycompsq)**

10.2 EXERCISES

Practice Makes Perfect

Complete the Square of a Binomial Expression

In the following exercises, complete the square to make a perfect square trinomial. Then, write the result as a binomial squared.

57. $a^2 + 10a$

58. $b^2 + 12b$

59. $m^2 + 18m$

60. $n^2 + 16n$

61. $m^2 - 24m$

62. $n^2 - 16n$

63. $p^2 - 22p$

64. $q^2 - 6q$

65. $x^2 - 9x$

66. $y^2 + 11y$

67. $p^2 - \frac{1}{3}p$

68. $q^2 + \frac{3}{4}q$

Solve Quadratic Equations of the Form $x^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

69. $v^2 + 6v = 40$

70. $w^2 + 8w = 65$

71. $u^2 + 2u = 3$

72. $z^2 + 12z = -11$

73. $c^2 - 12c = 13$

74. $d^2 - 8d = 9$

75. $x^2 - 20x = 21$

76. $y^2 - 2y = 8$

77. $m^2 + 4m = -44$

78. $n^2 - 2n = -3$

79. $r^2 + 6r = -11$

80. $t^2 - 14t = -50$

81. $a^2 - 10a = -5$

82. $b^2 + 6b = 41$

83. $u^2 - 14u + 12 = -1$

84. $z^2 + 2z - 5 = 2$

85. $v^2 = 9v + 2$

86. $w^2 = 5w - 1$

87. $(x + 6)(x - 2) = 9$

88. $(y + 9)(y + 7) = 79$

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

89. $3m^2 + 30m - 27 = 6$

90. $2n^2 + 4n - 26 = 0$

91. $2c^2 + c = 6$

92. $3d^2 - 4d = 15$

93. $2p^2 + 7p = 14$

94. $3q^2 - 5q = 9$

Everyday Math

95. Rafi is designing a rectangular playground to have an area of 320 square feet. He wants one side of the playground to be four feet longer than the other side. Solve the equation $p^2 + 4p = 320$ for p, the length of one side of the playground. What is the length of the other side?

96. Yvette wants to put a square swimming pool in the corner of her backyard. She will have a 3 foot deck on the south side of the pool and a 9 foot deck on the west side of the pool. She has a total area of 1080 square feet for the pool and two decks. Solve the equation $(s + 3)(s + 9) = 1080$ for s, the length of a side of the pool.

Writing Exercises

97. Solve the equation $x^2 + 10x = -25$ ⓐ by using the Square Root Property and ⓑ by completing the square. ⓒ Which method do you prefer? Why?

98. Solve the equation $y^2 + 8y = 48$ by completing the square and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
complete the square of a binomial expression.			
solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square.			
solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

10.3 Solve Quadratic Equations Using the Quadratic Formula

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the quadratic formula
- Use the discriminant to predict the number of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{-20 - 5}{10}$.
   If you missed this problem, review **Example 1.74**.
2. Simplify: $4 + \sqrt{121}$.
   If you missed this problem, review **Example 9.29**.
3. Simplify: $\sqrt{128}$.
   If you missed this problem, review **Example 9.12**.

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes.' In this section, we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general' so that we would do the algebraic steps only once and then use the new formula to find the value of the specific variable. Now, we will go through the steps of completing the square in general to solve a quadratic equation for *x*. It may be helpful to look at one of the examples at the end of the last section where we solved an equation of the form $ax^2 + bx + c = 0$ as you read through the algebraic steps below, so you see them with numbers as well as 'in general.'

We start with the standard form of a quadratic equation and solve it for x by completing the square.	$ax^2+bx+c = 0 \quad a \neq 0$
Isolate the variable terms on one side.	$ax^2+bx = -c$
Make leading coefficient 1, y dividing by a.	$\frac{ax^2}{a}+\frac{b}{a}x = -\frac{c}{a}$
Simplify.	$x^2+\frac{b}{a}x = -\frac{c}{a}$
To complete the square, fin $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ and add it to both sides of the equation. $\left(\frac{1}{2}\frac{b}{a}\right)^2=\frac{b^2}{4a^2}$	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2} = -\frac{c}{a}+\frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x+\frac{b}{2a}\right)^2 = -\frac{c}{a}+\frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}-\frac{c\cdot 4a}{a\cdot 4a}$
Simplify.	$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify.	$x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x = -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This last equation is the Quadratic Formula.

Quadratic Formula

The solutions to a quadratic equation of the form $ax^2+bx+c=0$, $a \neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of a, b, and c into the expression on the right side of the formula. Then, we do all the math to simplify the expression. The result gives the solution(s) to the quadratic equation.

EXAMPLE 10.28 HOW TO SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA

Solve $2x^2+9x-5=0$ by using the Quadratic Formula.

Solution

Step 1. Write the quadratic equation in standard form. Identify the a, b, c values.	This equation is in standard form.	$ax^2 + bx + c = 0$ $2x^2 + 9x - 5 = 0$ $a = 2, b = 9, c = -5$
Step 2. Write the quadratic formula. Then substitute in the values of a, b, c.	Substitute in $a = 2$, $b = 9$, $c = -5$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-9 \pm \sqrt{9^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2}$
Step 3. Simplify the fraction, and solve for x.		$x = \frac{-9 \pm \sqrt{81 - (-40)}}{4}$ $x = \frac{-9 \pm \sqrt{121}}{4}$ $x = \frac{-9 \pm 11}{4}$ $x = \frac{-9 + 11}{4}$ $\quad x = \frac{-9 - 11}{4}$ $x = \frac{2}{4}$ $\quad x = \frac{-20}{4}$ $x = \frac{1}{2}$ $\quad x = -5$
Step 4. Check the solutions.	Put each answer in the original equation to check. Substitute $x = \frac{1}{2}$. Substitute $x = -5$.	$2x^2 + 9x - 5 = 0$ $2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$ $\frac{10}{2} - 5 \stackrel{?}{=} 0$ $5 - 5 \stackrel{?}{=} 0$ $0 = 0 \checkmark$ $2x^2 + 9x - 5 = 0$ $2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$ $2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$ $50 - 45 - 5 \stackrel{?}{=} 0$ $0 = 0 \checkmark$

TRY IT : : 10.55 Solve $3y^2 - 5y + 2 = 0$ by using the Quadratic Formula.

TRY IT : : 10.56 Solve $4z^2 + 2z - 6 = 0$ by using the Quadratic Formula.

HOW TO : : SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA.

Step 1. Write the Quadratic Formula in standard form. Identify the a, b, and c values.

Step 2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.

Step 3. Simplify.

Step 4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time. And remember, the Quadratic Formula is an equation. Be sure you start with ' $x =$ '.

EXAMPLE 10.29

Solve $x^2 - 6x + 5 = 0$ by using the Quadratic Formula.

Solution

	$x^2 - 6x + 5 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $x^2 - 6x + 5 = 0$
Identify the *a, b, c* values.	$a = 1, b = -6, c = 5$
Write the Quadratic Formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a, b, c.*	$x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$
Simplify.	$x = \dfrac{6 \pm \sqrt{36 - 20}}{2}$ $x = \dfrac{6 \pm \sqrt{16}}{2}$ $x = \dfrac{6 \pm 4}{2}$
Rewrite to show two solutions.	$x = \dfrac{6+4}{2}, x = \dfrac{6-4}{2}$
Simplify.	$x = \dfrac{10}{2}, x = \dfrac{2}{2}$ $x = 5, x = 1$

Check.

$$\begin{array}{ll} x^2 - 6x + 5 = 0 & x^2 - 6x + 5 = 0 \\ 5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0 & 1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0 \\ 25 - 30 + 5 \stackrel{?}{=} 0 & 1 - 6 + 5 \stackrel{?}{=} 0 \\ 0 = 0 \checkmark & 0 = 0 \checkmark \end{array}$$

TRY IT : : 10.57 Solve $a^2 - 2a - 15 = 0$ by using the Quadratic Formula.

TRY IT : : 10.58 Solve $b^2 + 10b + 24 = 0$ by using the Quadratic Formula.

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, the final answer must have the radical in its simplified form.

EXAMPLE 10.30

Solve $4y^2 - 5y - 3 = 0$ by using the Quadratic Formula.

Solution

We can use the Quadratic Formula to solve for the variable in a quadratic equation, whether or not it is named '*x*'.

	$4y^2 - 5y - 3 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $4y^2 - 5y - 3 = 0$
Identify the *a*, *b*, *c* values.	$a = 4,\ b = -5,\ c = -3$
Write the Quadratic Formula.	$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 4 \cdot (-3)}}{2 \cdot 4}$
Simplify.	$y = \frac{5 \pm \sqrt{25 + 48}}{8}$ $y = \frac{5 \pm \sqrt{73}}{8}$
Rewrite to show two solutions.	$y = \frac{5 + \sqrt{73}}{8},\ y = \frac{5 - \sqrt{73}}{8}$
Check. We leave the check to you.	

TRY IT : : 10.59 Solve $2p^2 + 8p + 5 = 0$ by using the Quadratic Formula.

TRY IT : : 10.60 Solve $5q^2 - 11q + 3 = 0$ by using the Quadratic Formula.

EXAMPLE 10.31

Solve $2x^2 + 10x + 11 = 0$ by using the Quadratic Formula.

Solution

	$2x^2 + 10x + 11 = 0$
This equation is in standard form.	$ax^2 + bx + c = 0$ $2x^2 + 10x + 11 = 0$
Identify the *a*, *b*, *c* values.	$a = 2,\ b = 10,\ c = 11$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$x = \frac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$
Simplify.	$x = \frac{-10 \pm \sqrt{100 - 88}}{4}$ $x = \frac{-10 \pm \sqrt{12}}{4}$
Simplify the radical.	$x = \frac{-10 \pm 4\sqrt{3}}{4}$

Factor out the common factor in the numerator.	$x = \frac{2(-5 \pm 2\sqrt{3})}{4}$
Remove the common factors.	$x = \frac{-5 \pm 2\sqrt{3}}{2}$
Rewrite to show two solutions.	$x = \frac{-5 + 2\sqrt{3}}{2},\ x = \frac{-5 - 2\sqrt{3}}{2}$
Check. We leave the check to you.	

TRY IT : : 10.61 Solve $3m^2 + 12m + 7 = 0$ by using the Quadratic Formula.

TRY IT : : 10.62 Solve $5n^2 + 4n - 4 = 0$ by using the Quadratic Formula.

We cannot take the square root of a negative number. So, when we substitute a, b, and c into the Quadratic Formula, if the quantity inside the radical is negative, the quadratic equation has no real solution. We will see this in the next example.

EXAMPLE 10.32

Solve $3p^2 + 2p + 9 = 0$ by using the Quadratic Formula.

Solution

This equation is in standard form.	$\overset{ax^2}{3p^2} + \overset{bx}{2p} + \overset{c}{9} = \overset{0}{0}$
Identify the *a, b, c* values.	$a = 3,\ b = 2,\ c = 9$
Write the Quadratic Formula.	$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a, b, c.*	$p = \frac{-(2) \pm \sqrt{(2)^2 - 4 \cdot 3 \cdot (9)}}{2 \cdot 3}$
Simplify.	$p = \frac{-2 \pm \sqrt{4 - 108}}{6}$
Simplify the radical.	$p = \frac{-2 \pm \sqrt{-104}}{6}$
We cannot take the square root of a negative number.	There is no real solution.

TRY IT : : 10.63 Solve $4a^2 - 3a + 8 = 0$ by using the Quadratic Formula.

TRY IT : : 10.64 Solve $5b^2 + 2b + 4 = 0$ by using the Quadratic Formula.

The quadratic equations we have solved so far in this section were all written in standard form, $ax^2 + bx + c = 0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

EXAMPLE 10.33

Solve $x(x + 6) + 4 = 0$ by using the Quadratic Formula.

Solution

	$x(x+6)+4=0$
Distribute to get the equation in standard form.	$x^2+6x+4=0$
This equation is now in standard form.	$ax^2+bx+c=0$ $x^2+6x+4=0$
Identify the *a, b, c* values.	$a=1,\ b=6,\ c=4$
Write the Quadratic Formula.	$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of *a, b, c.*	$x=\dfrac{-(6)\pm\sqrt{(6)^2-4\cdot1\cdot(4)}}{2\cdot1}$
Simplify.	$x=\dfrac{-6\pm\sqrt{36-16}}{2}$
Simplify inside the radical.	$x=\dfrac{-6\pm\sqrt{20}}{2}$
Simplify the radical.	$x=\dfrac{-6\pm2\sqrt{5}}{2}$
Factor out the common factor in the numerator.	$x=\dfrac{2(-3\pm2\sqrt{5})}{2}$
Remove the common factors.	$x=-3\pm2\sqrt{5}$
Rewrite to show two solutions.	$x=-3+2\sqrt{5},\quad x=-3-2\sqrt{5}$
Check. We leave the check to you.	

TRY IT : : 10.65 Solve $x(x+2)-5=0$ by using the Quadratic Formula.

TRY IT : : 10.66 Solve $y(3y-1)-2=0$ by using the Quadratic Formula.

When we solved linear equations, if an equation had too many fractions we 'cleared the fractions' by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions—to solve. We can use the same strategy with quadratic equations.

EXAMPLE 10.34

Solve $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$ by using the Quadratic Formula.

Solution

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$
Multiply.	$3u^2+4u=2$
Subtract 2 to get the equation in standard form.	$ax^2+bx+c=0$ $3u^2+4u-2=0$
Identify the *a, b, c* values.	$a=3,\ b=4,\ c=-2$

Write the Quadratic Formula.	$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a, b, c.*	$u = \frac{-(4) \pm \sqrt{(4)^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3}$
Simplify.	$u = \frac{-4 \pm \sqrt{16 + 24}}{6}$ $u = \frac{-4 \pm \sqrt{40}}{6}$
Simplify the radical.	$u = \frac{-4 \pm 2\sqrt{10}}{6}$
Factor out the common factor in the numerator.	$u = \frac{2(-2 \pm \sqrt{10})}{6}$
Remove the common factors.	$u = \frac{-2 \pm \sqrt{10}}{3}$
Rewrite to show two solutions.	$u = \frac{-2 + \sqrt{10}}{3},\ u = \frac{-2 - \sqrt{10}}{3}$
Check. We leave the check to you.	

TRY IT : : 10.67 Solve $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$ by using the Quadratic Formula.

TRY IT : : 10.68 Solve $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{2}$ by using the Quadratic Formula.

Think about the equation $(x-3)^2 = 0$. We know from the Zero Products Principle that this equation has only one solution: $x = 3$.

We will see in the next example how using the Quadratic Formula to solve an equation with a perfect square also gives just one solution.

EXAMPLE 10.35

Solve $4x^2 - 20x = -25$ by using the Quadratic Formula.

Solution

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$\begin{array}{c} ax^2 + bx + c = 0 \\ 4x^2 - 20x + 25 = 0 \end{array}$
Identify the *a*, *b*, *c* values.	$a = 4,\ b = -20,\ c = 25$
Write the Quadratic Formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, *c*.	$x = \dfrac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \dfrac{20 \pm \sqrt{400 - 400}}{8}$ $x = \dfrac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \dfrac{20}{8}$
Simplify the fraction.	$x = \dfrac{5}{2}$
Check. We leave the check to you.	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square?

TRY IT : : 10.69 Solve $r^2 + 10r + 25 = 0$ by using the Quadratic Formula.

TRY IT : : 10.70 Solve $25t^2 - 40t = -16$ by using the Quadratic Formula.

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two solutions, sometimes one solution, sometimes no real solutions. Is there a way to predict the number of solutions to a quadratic equation without actually solving the equation?

Yes, the quantity inside the radical of the Quadratic Formula makes it easy for us to determine the number of solutions. This quantity is called the discriminant.

Discriminant

In the Quadratic Formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the **discriminant**.

Let's look at the discriminant of the equations in **Example 10.28**, **Example 10.32**, and **Example 10.35**, and the number of solutions to those quadratic equations.

	Quadratic Equation (in standard form)	Discriminant $b^2 - 4ac$	Sign of the Discriminant	Number of real solutions
Example 10.28	$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2(-5) = 121$	+	2
Example 10.35	$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \cdot 4 \cdot 25 = 0$	0	1
Example 10.32	$3p^2 + 2p + 9 = 0$	$2^2 - 4 \cdot 3 \cdot 9 = -104$	–	0

When the discriminant is **positive** $\left(x = \frac{-b \pm \sqrt{+}}{2a}\right)$ the quadratic equation has **two solutions**.

When the discriminant is **zero** $\left(x = \frac{-b \pm \sqrt{0}}{2a}\right)$ the quadratic equation has **one solution**.

When the discriminant is **negative** $\left(x = \frac{-b \pm \sqrt{-}}{2a}\right)$ the quadratic equation has **no real solutions**.

HOW TO : : USE THE DISCRIMINANT, $b^2 - 4ac$, TO DETERMINE THE NUMBER OF SOLUTIONS OF A QUADRATIC EQUATION.

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- if $b^2 - 4ac > 0$, the equation has two solutions.
- if $b^2 - 4ac = 0$, the equation has one solution.
- if $b^2 - 4ac < 0$, the equation has no real solutions.

EXAMPLE 10.36

Determine the number of solutions to each quadratic equation:

ⓐ $2v^2 - 3v + 6 = 0$ ⓑ $3x^2 + 7x - 9 = 0$ ⓒ $5n^2 + n + 4 = 0$ ⓓ $9y^2 - 6y + 1 = 0$

Solution

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

ⓐ

	$2v^2 - 3v + 6 = 0$
The equation is in standard form, identify a, b, c.	$a = 2,\ b = -3,\ c = 6$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(3)^2 - 4 \cdot 2 \cdot 6$
Simplify.	$9 - 48$ -39
Because the discriminant is negative, there are no real solutions to the equation.	

ⓑ

	$3x^2 + 7x - 9 = 0$
The equation is in standard form, identify a, b, c.	$a = 3,\ b = 7,\ c = -9$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(7)^2 - 4 \cdot 3 \cdot (-9)$
Simplify.	$49 + 108$ 157
Because the discriminant is positive, there are two solutions to the equation.	

ⓒ

	$5n^2 + n + 4 = 0$
The equation is in standard form, identify a, b, and c.	$a = 5,\ b = 1,\ c = 4$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(1)^2 - 4 \cdot 5 \cdot 4$
Simplify.	$1 - 80$ -79

Because the discriminant is negative, there are no real solutions to the equation.

ⓓ

	$9y^2 - 6y + 1 = 0$
The equation is in standard form, identify a, b, c.	$a = 9,\ b = -6,\ c = 1$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of a, b, c.	$(-6)^2 - 4 \cdot 9 \cdot 1$
Simplify.	$36 - 36$ 0

Because the discriminant is 0, there is one solution to the equation.

TRY IT : : 10.71

Determine the number of solutions to each quadratic equation:

ⓐ $8m^2 - 3m + 6 = 0$ ⓑ $5z^2 + 6z - 2 = 0$ ⓒ $9w^2 + 24w + 16 = 0$ ⓓ $9u^2 - 2u + 4 = 0$

TRY IT : : 10.72

Determine the number of solutions to each quadratic equation:

ⓐ $b^2 + 7b - 13 = 0$ ⓑ $5a^2 - 6a + 10 = 0$ ⓒ $4r^2 - 20r + 25 = 0$ ⓓ $7t^2 - 11t + 3 = 0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We have used four methods to solve quadratic equations:

- Factoring
- Square Root Property
- Completing the Square
- Quadratic Formula

You can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method to use.

HOW TO : : IDENTIFY THE MOST APPROPRIATE METHOD TO SOLVE A QUADRATIC EQUATION.

Step 1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.

Step 2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x - h)^2 = k$, it can easily be solved by using the Square Root Property.

Step 3. Use the **Quadratic Formula**. Any quadratic equation can be solved by using the Quadratic Formula.

What about the method of completing the square? Most people find that method cumbersome and prefer not to use it. We needed to include it in this chapter because we completed the square in general to derive the Quadratic Formula. You will also use the process of completing the square in other areas of algebra.

EXAMPLE 10.37

Identify the most appropriate method to use to solve each quadratic equation:

ⓐ $5z^2 = 17$ ⓑ $4x^2 - 12x + 9 = 0$ ⓒ $8u^2 + 6u = 11$

Solution

ⓐ $5z^2 = 17$

Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

ⓑ $4x^2 - 12x + 9 = 0$

We recognize that the left side of the equation is a perfect square trinomial, and so Factoring will be the most appropriate method.

ⓒ $8u^2 + 6u = 11$

Put the equation in standard form. $8u^2 + 6u - 11 = 0$

While our first thought may be to try Factoring, thinking about all the possibilities for trial and error leads us to choose the Quadratic Formula as the most appropriate method

TRY IT : : 10.73 Identify the most appropriate method to use to solve each quadratic equation:

ⓐ $x^2 + 6x + 8 = 0$ ⓑ $(n - 3)^2 = 16$ ⓒ $5p^2 - 6p = 9$

TRY IT : : 10.74 Identify the most appropriate method to use to solve each quadratic equation:

ⓐ $8a^2 + 3a - 9 = 0$ ⓑ $4b^2 + 4b + 1 = 0$ ⓒ $5c^2 = 125$

MEDIA : :

Access these online resources for additional instruction and practice with using the Quadratic Formula:

- **Solving Quadratic Equations: Solving with the Quadratic Formula (https://openstax.org/l/25Quadformula)**
- **How to solve a quadratic equation in standard form using the Quadratic Formula (example) (https://openstax.org/l/25Usequadform)**
- **Solving Quadratic Equations using the Quadratic Formula—Example 3 (https://openstax.org/l/25Byquadformula)**
- **Solve Quadratic Equations using Quadratic Formula (https://openstax.org/l/25Solvequadform)**

10.3 EXERCISES

Practice Makes Perfect

Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

99. $4m^2 + m - 3 = 0$

100. $4n^2 - 9n + 5 = 0$

101. $2p^2 - 7p + 3 = 0$

102. $3q^2 + 8q - 3 = 0$

103. $p^2 + 7p + 12 = 0$

104. $q^2 + 3q - 18 = 0$

105. $r^2 - 8r - 33 = 0$

106. $t^2 + 13t + 40 = 0$

107. $3u^2 + 7u - 2 = 0$

108. $6z^2 - 9z + 1 = 0$

109. $2a^2 - 6a + 3 = 0$

110. $5b^2 + 2b - 4 = 0$

111. $2x^2 + 3x + 9 = 0$

112. $6y^2 - 5y + 2 = 0$

113. $v(v + 5) - 10 = 0$

114. $3w(w - 2) - 8 = 0$

115. $\frac{1}{3}m^2 + \frac{1}{12}m = \frac{1}{4}$

116. $\frac{1}{3}n^2 + n = -\frac{1}{2}$

117. $16c^2 + 24c + 9 = 0$

118. $25d^2 - 60d + 36 = 0$

119. $5m^2 + 2m - 7 = 0$

120. $8n^2 - 3n + 3 = 0$

121. $p^2 - 6p - 27 = 0$

122. $25q^2 + 30q + 9 = 0$

123. $4r^2 + 3r - 5 = 0$

124. $3t(t - 2) = 2$

125. $2a^2 + 12a + 5 = 0$

126. $4d^2 - 7d + 2 = 0$

127. $\frac{3}{4}b^2 + \frac{1}{2}b = \frac{3}{8}$

128. $\frac{1}{9}c^2 + \frac{2}{3}c = 3$

129. $2x^2 + 12x - 3 = 0$

130. $16y^2 + 8y + 1 = 0$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions to each quadratic equation.

131.
ⓐ $4x^2 - 5x + 16 = 0$
ⓑ $36y^2 + 36y + 9 = 0$
ⓒ $6m^2 + 3m - 5 = 0$
ⓓ $18n^2 - 7n + 3 = 0$

132.
ⓐ $9v^2 - 15v + 25 = 0$
ⓑ $100w^2 + 60w + 9 = 0$
ⓒ $5c^2 + 7c - 10 = 0$
ⓓ $15d^2 - 4d + 8 = 0$

133.
ⓐ $r^2 + 12r + 36 = 0$
ⓑ $8t^2 - 11t + 5 = 0$
ⓒ $4u^2 - 12u + 9 = 0$
ⓓ $3v^2 - 5v - 1 = 0$

134.
ⓐ $25p^2 + 10p + 1 = 0$
ⓑ $7q^2 - 3q - 6 = 0$
ⓒ $7y^2 + 2y + 8 = 0$
ⓓ $25z^2 - 60z + 36 = 0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

135.
ⓐ $x^2 - 5x - 24 = 0$
ⓑ $(y + 5)^2 = 12$
ⓒ $14m^2 + 3m = 11$

136.
ⓐ $(8v + 3)^2 = 81$
ⓑ $w^2 - 9w - 22 = 0$
ⓒ $4n^2 - 10 = 6$

137.
ⓐ $6a^2 + 14 = 20$
ⓑ $\left(x - \frac{1}{4}\right)^2 = \frac{5}{16}$
ⓒ $y^2 - 2y = 8$

138.
ⓐ $8b^2 + 15b = 4$
ⓑ $\frac{5}{9}v^2 - \frac{2}{3}v = 1$
ⓒ $\left(w + \frac{4}{3}\right)^2 = \frac{2}{9}$

Everyday Math

139. A flare is fired straight up from a ship at sea. Solve the equation $16\left(t^2 - 13t + 40\right) = 0$ for t, the number of seconds it will take for the flare to be at an altitude of 640 feet.

140. An architect is designing a hotel lobby. She wants to have a triangular window looking out to an atrium, with the width of the window 6 feet more than the height. Due to energy restrictions, the area of the window must be 140 square feet. Solve the equation $\frac{1}{2}h^2 + 3h = 140$ for h, the height of the window.

Writing Exercises

141. Solve the equation $x^2 + 10x = 200$
ⓐ by completing the square
ⓑ using the Quadratic Formula
ⓒ Which method do you prefer? Why?

142. Solve the equation $12y^2 + 23y = 24$
ⓐ by completing the square
ⓑ using the Quadratic Formula
ⓒ Which method do you prefer? Why?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
solve quadratic equations using the quadratic formula.			
use the discriminant to predict the number of solutions of a quadratic equation.			
identify the most appropriate method to use to solve a quadratic equation.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

10.4 Solve Applications Modeled by Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve applications modeled by Quadratic Equations

Be Prepared!

Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is -100. Find the numbers.
   If you missed this problem, review **Example 3.10**.
2. The area of triangular mural is 64 square feet. The base is 16 feet. Find the height.
   If you missed this problem, review **Example 3.36**.
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.
   If you missed this problem, review **Example 3.39**.

Solve Applications of the Quadratic Formula

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications. To get us started, we will copy our usual Problem Solving Strategy here so we can follow the steps.

HOW TO : : USE THE PROBLEM SOLVING STRATEGY.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

We have solved number applications that involved **consecutive even integers** and **consecutive odd integers** by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

	Consecutive even integers		**Consecutive odd integers**
	64, 66, 68		77, 79, 81
n	1^{st} even integer	n	1^{st} odd integer
$n+2$	2^{nd} consecutive even integer	$n+2$	2^{nd} consecutive odd integer
$n+4$	3^{rd} consecutive even integer	$n+4$	3^{rd} consecutive odd integer

Some applications of consecutive odd integers or consecutive even integers are modeled by quadratic equations. The notation above will be helpful as you name the variables.

EXAMPLE 10.38

The product of two consecutive odd integers is 195. Find the integers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive odd integers.
Step 3. Name what we are looking for.	Let $n =$ the first odd integer. $n + 2 =$ the next odd integer
Step 4. Translate into an equation. State the problem in one sentence.	"The product of two consecutive odd integers is 195." The product of the first odd integer and the second odd integer is 195.
Translate into an equation	$n(n+2) = 195$
Step 5. Solve the equation. Distribute.	$n^2 + 2n = 195$
Subtract 195 to get the equation in standard form.	$ax^2 + bx + c = 0$ $n^2 + 2n - 195 = 0$
Identify the *a, b, c* values.	$a = 1,\ b = 2,\ c = -195$
Write the quadratic equation.	$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a, b, c*..	$n = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-195)}}{2 \cdot 1}$
Simplify.	$n = \frac{-2 \pm \sqrt{4 + 780}}{2}$ $n = \frac{-2 \pm \sqrt{784}}{2}$
Simplify the radical.	$n = \frac{-2 \pm 28}{2}$
Rewrite to show two solutions.	$n = \frac{-2 + 28}{2},\ n = \frac{-2 - 28}{2}$
Solve each equation.	$n = \frac{26}{2},\ n = \frac{-30}{2}$ $n = 13 \quad n = -15$
There are two values of *n* that are solutions. This will give us two pairs of consecutive odd integers for our solution.	First odd integer $n = 13$ next odd integer $n + 2$ $13 + 2$ 15
	First odd integer $n = -15$ next odd integer $n + 2$ $-15 + 2$ -13
Step 6. Check the answer. Do these pairs work? Are they consecutive odd integers? Is their product 195?	$13, 15,$ yes $\quad -13, -15,$ yes $13 \cdot 15 = 195,$ yes $\quad -13(-15) = 195,$ yes
Step 7. Answer the question.	The two consecutive odd integers whose product is 195 are 13, 15, and −13, −15.

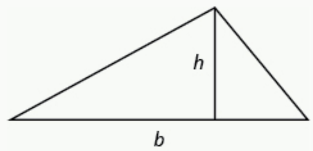
TRY IT : : 10.75 The product of two consecutive odd integers is 99. Find the integers.

TRY IT : : 10.76 The product of two consecutive odd integers is 168. Find the integers.

We will use the formula for the area of a triangle to solve the next example.

Area of a Triangle

For a triangle with base b and height h, the area, A, is given by the formula $A = \frac{1}{2}bh$.

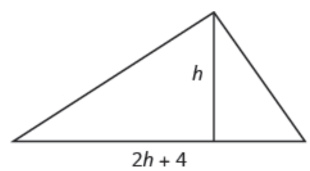


Recall that, when we solve geometry applications, it is helpful to draw the figure.

EXAMPLE 10.39

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can have an area of 120 square feet and the architect wants the width to be 4 feet more than twice the height. Find the height and width of the window.

Solution

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the height and width.
Step 3. Name what we are looking for.	Let $h =$ the height of the triangle. $2h + 4 =$ the width of the triangle
Step 4. Translate.	We know the area. Write the formula for the area of a triangle. $A = \frac{1}{2}bh$
Step 5. Solve the equation. Substitute in the values.	$120 = \frac{1}{2}(2h + 4)h$
Distribute.	$120 = h^2 + 2h$
This is a quadratic equation, rewrite it in standard form.	$ax^2 + bx + c = 0$ $h^2 + 2h - 120 = 0$
Solve the equation using the Quadratic Formula. Identify the *a, b, c* values.	$a = 1, b = 2, c = -120$
Write the quadratic equation.	$h = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a, b, c*..	$h = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-120)}}{2 \cdot 1}$

Simplify.	$h = \frac{-2 \pm \sqrt{4 + 480}}{2}$ $h = \frac{-2 \pm \sqrt{484}}{2}$
Simplify the radical.	$h = \frac{-2 \pm 22}{2}$
Rewrite to show two solutions.	$h = \frac{-2 + 22}{2}, \; h = \frac{-2 - 22}{2}$
Simplify.	$h = \frac{20}{2}, \; h = \frac{-24}{2}$
Since *h* is the height of a window, a value of $h = -12$ does not make sense.	$h = 10 \quad \cancel{h = -12}$
	The height of the triangle: $h = 10$ The width of the triangle: $2h + 4$ $2 \cdot 10 + 4$ 24
Step 6. Check the answer. Does a triangle with a height 10 and width 24 have area 120? Yes.	
Step 7. Answer the question.	The height of the triangular window is 10 feet and the width is 24 feet.

Notice that the solutions were integers. That tells us that we could have solved the equation by factoring.

When we wrote the equation in standard form, $h^2 + 2h - 120 = 0$, we could have factored it. If we did, we would have solved the equation $(h + 12)(h - 10) = 0$.

TRY IT : : 10.77

Find the dimensions of a triangle whose width is four more than six times its height and has an area of 208 square inches.

TRY IT : : 10.78

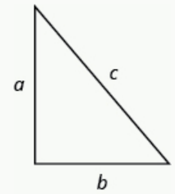
If a triangle that has an area of 110 square feet has a height that is two feet less than twice the width, what are its dimensions?

In the two preceding examples, the number in the radical in the Quadratic Formula was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

The Pythagorean Theorem gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Pythagorean Theorem

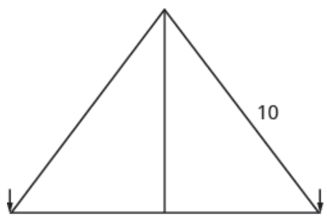
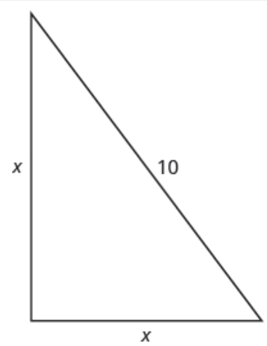
In any right triangle, where a and b are the lengths of the legs and c is the length of the hypotenuse, $a^2 + b^2 = c^2$



EXAMPLE 10.40

Rene is setting up a holiday light display. He wants to make a ‘tree’ in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Solution

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the height of the pole.
Step 3. Name what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the pole. Let $x =$ the height of the pole. $x =$ the distance from the pole to stake
Each side is a right triangle. We draw a picture of one of them.	
Step 4. Translate into an equation. We can use the Pythagorean Theorem to solve for *x*.	
Write the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Step 5. Solve the equation. Substitute.	$x^2 + x^2 = 10^2$
Simplify.	$2x^2 = 100$
Divide by 2 to isolate the variable.	$\frac{2x^2}{2} = \frac{100}{2}$
Simplify.	$x^2 = 50$

Use the Square Root Property.	$x = \pm\sqrt{50}$
Simplify the radical.	$x = \pm 5\sqrt{2}$
Rewrite to show two solutions.	$x = 5\sqrt{2}$ $\cancel{x = -5\sqrt{2}}$
Approximate this number to the nearest tenth with a calculator.	$x \approx 7.1$
Step 6. Check the answer. Check on your own in the Pythagorean Theorem.	
Step 7. Answer the question.	The pole should be about 7.1 feet tall.

TRY IT : : 10.79

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth of a foot.

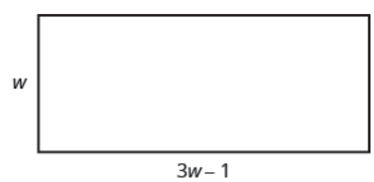
TRY IT : : 10.80

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

EXAMPLE 10.41

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than three times the width. Find the length and width. Round to the nearest tenth of a foot.

Solution

Step 1. Read the problem. Draw a picture.	
Step 2. Identify what we are looking for.	We are looking for the length and width.
Step 3. Name what we are looking for.	Let $w =$ the width of the rectangle. $3w - 1 =$ the length of the rectangle
Step 4. Translate into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$
Step 5. Solve the equation. Substitute in the values.	$150 = (3w - 1)w$
Distribute.	$150 = 3w^2 - w$

This is a quadratic equation, rewrite it in standard form.	$\overset{ax^2}{3w^2} \overset{+bx}{-w} \overset{+c}{-150} \overset{=0}{=0}$
Solve the equation using the Quadratic Formula.	
Identify the *a, b, c* values.	$a=3,\ b=-1,\ c=-150$
Write the Quadratic Formula.	$w=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of *a, b, c*.	$w=\frac{-(-1)\pm\sqrt{(-1)^2-4\cdot 3\cdot(-150)}}{2\cdot 3}$
Simplify.	$w=\frac{1\pm\sqrt{1+1800}}{6}$ $w=\frac{1\pm\sqrt{1801}}{6}$
Rewrite to show two solutions.	$w=\frac{1+\sqrt{1801}}{6},\ w=\frac{1-\sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	$w\approx 7.2,\quad \cancel{w\approx -6.9}$ Width $w\approx 7.2$ Length $\approx 3w-1$ $\approx 3(7.2)-1$ ≈ 20.6
Step 6. Check the answer. Make sure that the answers make sense.	
Step 7. Answer the question.	The width of the rectangle is approximately 7.2 feet and the length 20.6 feet.

TRY IT : : 10.81

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden. Round to the nearest tenth of a foot.

TRY IT : : 10.82

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length. What are the length and width of the tablecloth? Round to the nearest tenth of a foot.

The height of a projectile shot upwards is modeled by a quadratic equation. The initial velocity, v_0, propels the object up until gravity causes the object to fall back down.

Projectile Motion

The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula:

$$h=-16t^2+v_0t$$

We can use the formula for projectile motion to find how many seconds it will take for a firework to reach a specific height.

EXAMPLE 10.42

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for the number of seconds, which is time.
Step 3. Name what we are looking for.	Let $t =$ the number of seconds.
Step 4. Translate into an equation.	Use the formula.
	$h = -16t^2 + v_0 t$
Step 5. Solve the equation. We know the velocity v_0 is 130 feet per second.	
The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$
This is a quadratic equation, rewrite it in standard form.	$\begin{array}{l} ax^2 + bx + c = 0 \\ 16t^2 + 130t + 260 = 0 \end{array}$
Solve the equation using the Quadratic Formula.	
Identify the *a, b, c* values.	$a = 16, b = -130, c = 260$
Write the Quadratic Formula.	$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a, b, c*.	$t = \frac{-(-130) \pm \sqrt{(-130)^2 - 4 \cdot 16 \cdot (260)}}{2 \cdot 16}$
Simplify.	$t = \frac{130 \pm \sqrt{16{,}900 - 16{,}640}}{32}$ $t = \frac{130 \pm \sqrt{260}}{32}$
Rewrite to show two solutions.	$t = \frac{130 + \sqrt{260}}{32}, \ t = \frac{130 - \sqrt{260}}{32}$
Approximate the answers with a calculator.	$t \approx 4.6$ seconds, $t \approx 3.6$
Step 6. Check the answer. The check is left to you.	
Step 7. Answer the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds. 

TRY IT : : 10.83

An arrow is shot from the ground into the air at an initial speed of 108 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth of a second.

TRY IT : : 10.84

A man throws a ball into the air with a velocity of 96 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth of a second.

MEDIA : :

Access these online resources for additional instruction and practice with solving word problems using the quadratic equation:

- **General Quadratic Word Problems (https://openstax.org/l/25Quadproblem)**
- **Word problem: Solve a projectile problem using a quadratic equation (https://openstax.org/l/25Projectile)**

10.4 EXERCISES

Practice Makes Perfect

Solve Applications of the Quadratic Formula

In the following exercises, solve by using methods of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth.

143. The product of two consecutive odd numbers is 255. Find the numbers.

144. The product of two consecutive even numbers is 360. Find the numbers.

145. The product of two consecutive even numbers is 624. Find the numbers.

146. The product of two consecutive odd numbers is 1023. Find the numbers.

147. The product of two consecutive odd numbers is 483. Find the numbers.

148. The product of two consecutive even numbers is 528. Find the numbers.

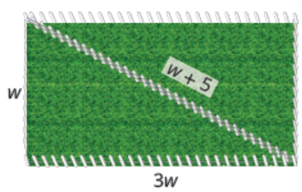
149. A triangle with area 45 square inches has a height that is two less than four times the width. Find the height and width of the triangle.

150. The width of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the height and width of the triangle.

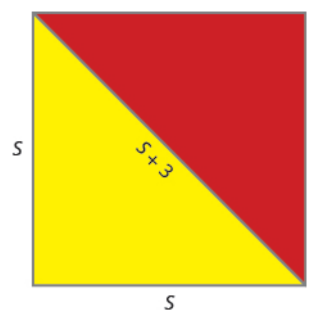
151. The hypotenuse of a right triangle is twice the length of one of its legs. The length of the other leg is three feet. Find the lengths of the three sides of the triangle.

152. The hypotenuse of a right triangle is 10 cm long. One of the triangle's legs is three times the length of the other leg. Round to the nearest tenth. Find the lengths of the three sides of the triangle.

153. A farmer plans to fence off sections of a rectangular corral. The diagonal distance from one corner of the corral to the opposite corner is five yards longer than the width of the corral. The length of the corral is three times the width. Find the length of the diagonal of the corral.



154. Nautical flags are used to represent letters of the alphabet. The flag for the letter O consists of a yellow right triangle and a red right triangle which are sewn together along their hypotenuse to form a square. The adjoining side of the two triangles is three inches longer than a side of the flag. Find the length of the side of the flag.



155. The length of a rectangular driveway is five feet more than three times the width. The area is 350 square feet. Find the length and width of the driveway.

156. A rectangular lawn has area 140 square yards. Its width that is six less than twice the length. What are the length and width of the lawn?

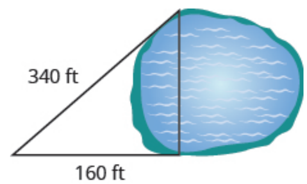
157. A firework rocket is shot upward at a rate of 640 ft/sec. Use the projectile formula $h = -16t^2 + v_0 t$ to determine when the height of the firework rocket will be 1200 feet.

158. An arrow is shot vertically upward at a rate of 220 feet per second. Use the projectile formula $h = -16t^2 + v_0 t$ to determine when height of the arrow will be 400 feet.

Everyday Math

159. A bullet is fired straight up from a BB gun with initial velocity 1120 feet per second at an initial height of 8 feet. Use the formula $h = -16t^2 + v_0 t + 8$ to determine how many seconds it will take for the bullet to hit the ground. (That is, when will $h = 0$?)

160. A city planner wants to build a bridge across a lake in a park. To find the length of the bridge, he makes a right triangle with one leg and the hypotenuse on land and the bridge as the other leg. The length of the hypotenuse is 340 feet and the leg is 160 feet. Find the length of the bridge.



Writing Exercises

161. Make up a problem involving the product of two consecutive odd integers. Start by choosing two consecutive odd integers. ⓐ What are your integers? ⓑ What is the product of your integers? ⓒ Solve the equation $n(n+2) = p$, where p is the product you found in part (b). ⓓ Did you get the numbers you started with?

162. Make up a problem involving the product of two consecutive even integers. Start by choosing two consecutive even integers. ⓐ What are your integers? ⓑ What is the product of your integers? ⓒ Solve the equation $n(n+2) = p$, where p is the product you found in part (b). ⓓ Did you get the numbers you started with?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve applications of the quadratic formula.			

ⓑ *On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

10.5 Graphing Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Recognize the graph of a quadratic equation in two variables
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic equations in two variables
- Solve maximum and minimum applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Graph the equation $y = 3x - 5$ by plotting points.
   If you missed this problem, review **Example 4.11**.
2. Evaluate $2x^2 + 4x - 1$ when $x = -3$.
   If you missed this problem, review **Example 1.57**.
3. Evaluate $-\frac{b}{2a}$ when $a = \frac{1}{3}$ and $b = \frac{5}{6}$.
   If you missed this problem, review **Example 1.89**.

Recognize the Graph of a Quadratic Equation in Two Variables

We have graphed equations of the form $Ax + By = C$. We called equations like this linear equations because their graphs are straight lines.

Now, we will graph equations of the form $y = ax^2 + bx + c$. We call this kind of equation a quadratic equation in two variables.

Quadratic Equation in Two Variables

A **quadratic equation in two variables**, where $a, b,$ and c are real numbers and $a \neq 0$, is an equation of the form

$$y = ax^2 + bx + c$$

Just like we started graphing linear equations by plotting points, we will do the same for quadratic equations.

Let's look first at graphing the quadratic equation $y = x^2$. We will choose integer values of x between -2 and 2 and find their y values. See **Table 10.1**.

$y = x^2$	
x	y
0	0
1	1
-1	1
2	4
-2	4

Table 10.1

Notice when we let $x = 1$ and $x = -1$, we got the same value for y.

$$\begin{array}{ll} y = x^2 & y = x^2 \\ y = 1^2 & y = (-1)^2 \\ y = 1 & y = 1 \end{array}$$

The same thing happened when we let $x = 2$ and $x = -2$.

Now, we will plot the points to show the graph of $y = x^2$. See **Figure 10.2**.

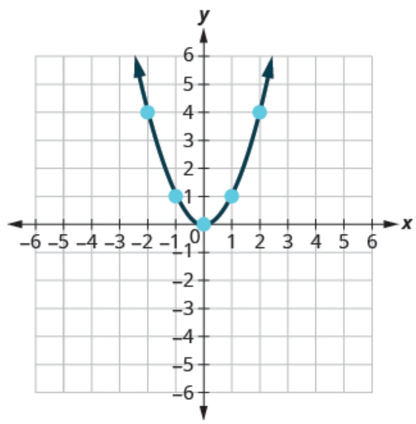


Figure 10.2

The graph is not a line. This figure is called a **parabola**. Every quadratic equation has a graph that looks like this.
In **Example 10.43** you will practice graphing a parabola by plotting a few points.

EXAMPLE 10.43

Graph $y = x^2 - 1$.

Solution

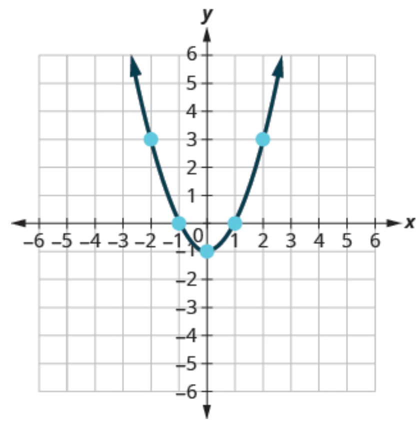
We will graph the equation by plotting points.

Choose integers values for *x*, substitute them into the equation and solve for *y*.

Record the values of the ordered pairs in the chart.

$y = x^2 - 1$	
x	y
0	−1
1	0
−1	0
2	3
−2	3

Plot the points, and then connect them with a smooth curve. The result will be the graph of the equation $y = x^2 - 1$.



TRY IT : : 10.85 Graph $y = -x^2$.

TRY IT : : 10.86 Graph $y = x^2 + 1$.

How do the equations $y = x^2$ and $y = x^2 - 1$ differ? What is the difference between their graphs? How are their graphs the same?

All parabolas of the form $y = ax^2 + bx + c$ open upwards or downwards. See **Figure 10.3**.

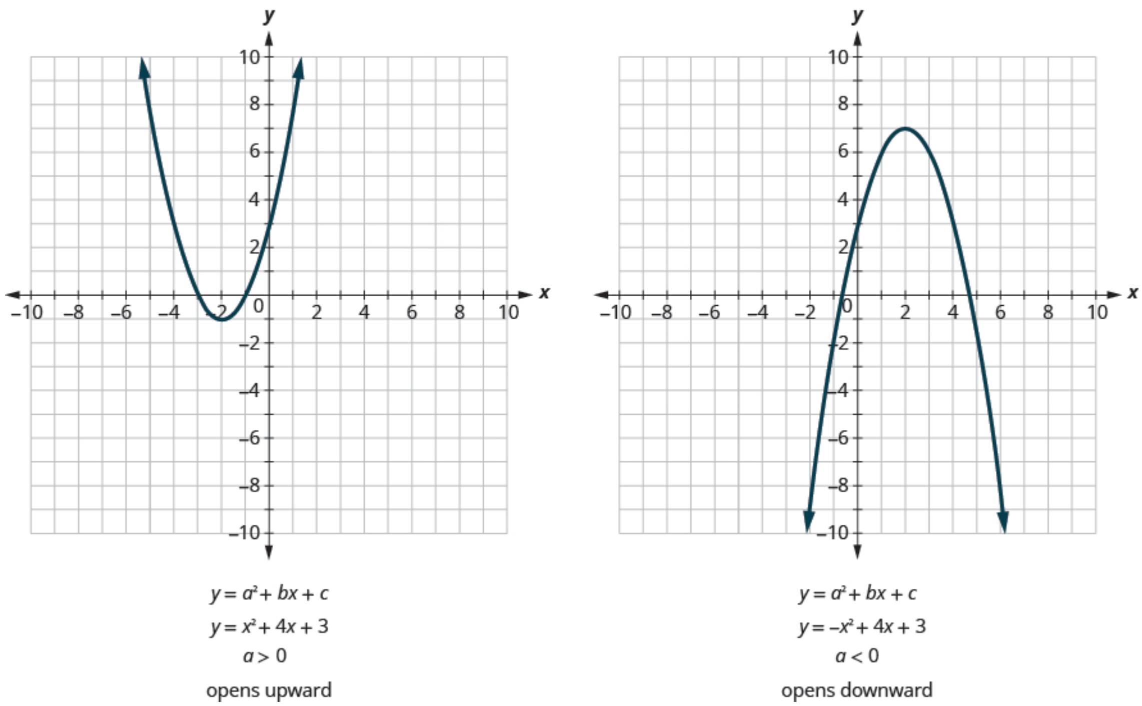


Figure 10.3

Notice that the only difference in the two equations is the negative sign before the x^2 in the equation of the second graph in **Figure 10.3**. When the x^2 term is positive, the parabola opens upward, and when the x^2 term is negative, the parabola opens downward.

Parabola Orientation

For the quadratic equation $y = ax^2 + bx + c$, if:

- $a > 0$, the parabola opens upward
- $a < 0$, the parabola opens downward

EXAMPLE 10.44

Determine whether each parabola opens upward or downward:

ⓐ $y = -3x^2 + 2x - 4$ ⓑ $y = 6x^2 + 7x - 9$

Solution

ⓐ Find the value of "*a*".	$y = \quad ax^2 + bx + c$ $y = -3x^2 + 2x - 4$ $a = -3$ Since the “a” is negative, the parabola will open downward.
ⓑ Find the value of "*a*".	$y = ax^2 + bx + c$ $y = 6x^2 + 7x - 9$ $a = 6$ Since the “a” is positive, the parabola will open upward.

TRY IT : : 10.87 Determine whether each parabola opens upward or downward:

ⓐ $y = 2x^2 + 5x - 2$ ⓑ $y = -3x^2 - 4x + 7$

TRY IT : : 10.88 Determine whether each parabola opens upward or downward:

ⓐ $y = -2x^2 - 2x - 3$ ⓑ $y = 5x^2 - 2x - 1$

Find the Axis of Symmetry and Vertex of a Parabola

Look again at Figure 10.3. Do you see that we could fold each parabola in half and that one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry in red. See Figure 10.4.

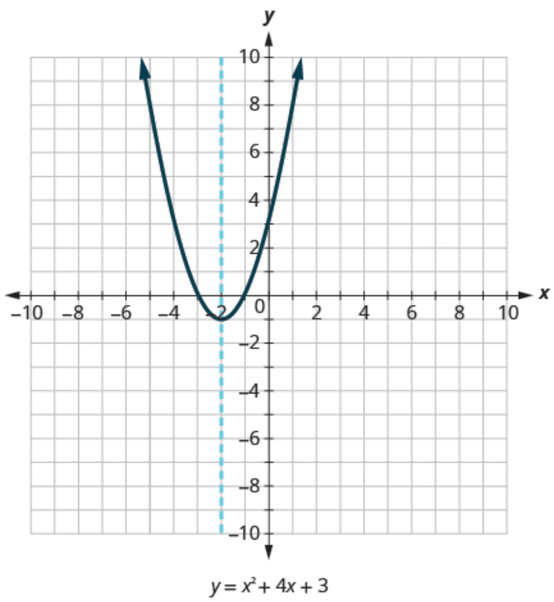


Figure 10.4

The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.

So, to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x = -\frac{b}{2a}$.

$\overset{ax^2 + bx + c}{y = x^2 + 4x + 3}$	$\overset{ax^2 + bx + c}{y = -x^2 + 4x + 3}$
axis of symmetry	axis of symmetry
$x = -\frac{b}{2a}$	$x = -\frac{b}{2a}$
$x = -\frac{4}{2 \cdot 1}$	$x = -\frac{4}{2(-1)}$
$x = -2$	$x = 2$

Look back at Figure 10.4. Are these the equations of the dashed red lines?

The point on the parabola that is on the axis of symmetry is the lowest or highest point on the parabola, depending on whether the parabola opens upwards or downwards. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its x-coordinate is $-\frac{b}{2a}$. To find the y-coordinate of the vertex, we substitute the value of the x-coordinate into the quadratic equation.

$y = x^2 + 4x + 3$	$y = -x^2 + 4x + 3$
axis of symmetry is $x = -2$	axis of symmetry is $x = 2$
vertex is $(-2, \_\_)$	vertex is $(2, \_\_)$
$y = x^2 + 4x + 3$ $y = (-2)^2 + 4(-2) + 3$ $y = -1$	$y = -x^2 + 4x + 3$ $y = -(2)^2 + 4(2) + 3$ $y = 7$
vertex is $(-2, -1)$	vertex is $(2, 7)$

Axis of Symmetry and Vertex of a Parabola

For a parabola with equation $y = ax^2 + bx + c$:

- The axis of symmetry of a parabola is the line $x = -\frac{b}{2a}$.
- The vertex is on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.

To find the *y*-coordinate of the vertex, we substitute $x = -\frac{b}{2a}$ into the quadratic equation.

EXAMPLE 10.45

For the parabola $y = 3x^2 - 6x + 2$ find: ⓐ the axis of symmetry and ⓑ the vertex.

Solution

ⓐ	$y = ax^2 + bx + c$ $y = 3x^2 - 6x + 2$
The axis of symmetry is the line $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
Substitute the values of *a, b* into the equation.	$x = -\frac{-6}{2 \cdot 3}$
Simplify.	$x = 1$
	The axis of symmetry is the line $x = 1$.
ⓑ	$y = 3x^2 - 6x + 2$
The vertex is on the line of symmetry, so its *x*-coordinate will be $x = 1$.	
Substitute $x = 1$ into the equation and solve for *y*.	$y = 3(1)^2 - 6(1) + 2$
Simplify.	$y = 3 \cdot 1 - 6 + 2$
This is the *y*-coordinate.	$y = -1$ The vertex is $(1, -1)$.

TRY IT : : 10.89 For the parabola $y = 2x^2 - 8x + 1$ find: ⓐ the axis of symmetry and ⓑ the vertex.

TRY IT : : 10.90 For the parabola $y = 2x^2 - 4x - 3$ find: ⓐ the axis of symmetry and ⓑ the vertex.

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the *x*- and *y*-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the ***y*-intercept** the value of x is zero. So, to find the *y*-intercept, we substitute $x = 0$ into the equation.

Let's find the *y*-intercepts of the two parabolas shown in the figure below.

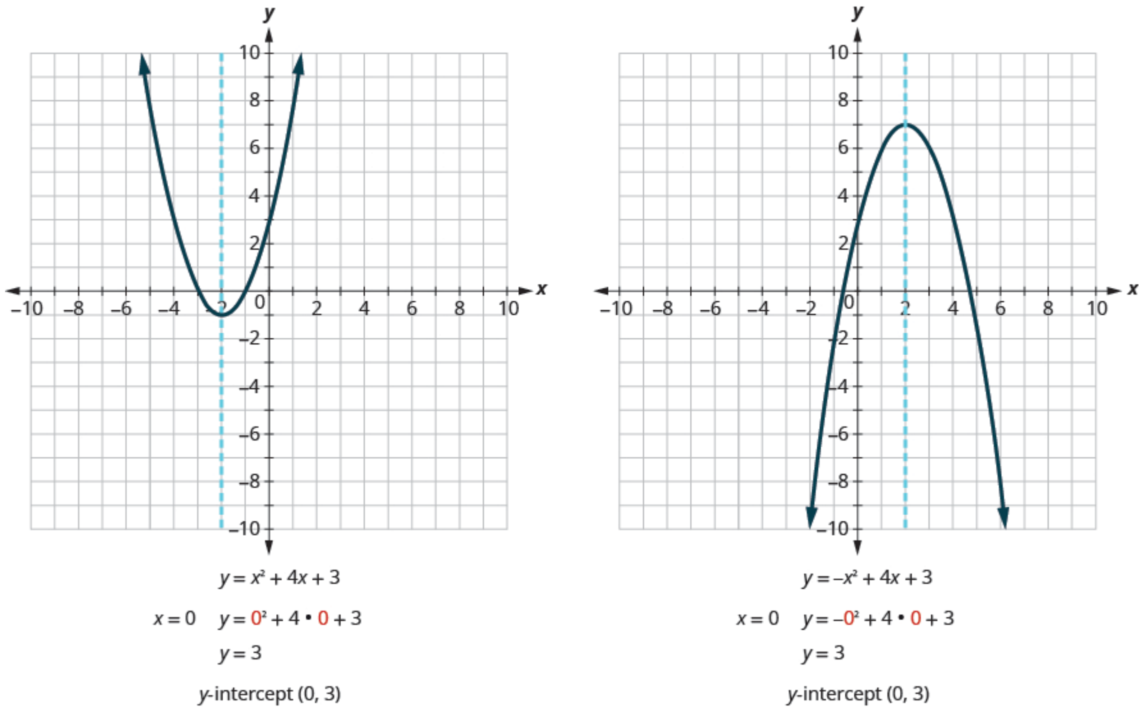


Figure 10.5

At an ***x*-intercept**, the value of y is zero. To find an *x*-intercept, we substitute $y = 0$ into the equation. In other words, we will need to solve the equation $0 = ax^2 + bx + c$ for x.

$$y = ax^2 + bx + c$$
$$0 = ax^2 + bx + c$$

But solving quadratic equations like this is exactly what we have done earlier in this chapter.

We can now find the *x*-intercepts of the two parabolas shown in **Figure 10.5**.

First, we will find the *x*-intercepts of a parabola with equation $y = x^2 + 4x + 3$.

	$y = x^2 + 4x + 3$
Let $y = 0$.	$0 = x^2 + 4x + 3$
Factor.	$0 = (x + 1)(x + 3)$
Use the zero product property.	$x + 1 = 0,\ x + 3 = 0$
Solve.	$x = -1,\ x = -3$
	The *x* intercepts are $(-1, 0)$ and $(-3, 0)$.

Now, we will find the *x*-intercepts of the parabola with equation $y = -x^2 + 4x + 3$.

	$y = -x^2 + 4x + 3$
Let $y = 0$.	$0 = -x^2 + 4x + 3$
This quadratic does not factor, so we use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$a = -1$, $b = 4$, $c = 3$	$x = \frac{-4 \pm \sqrt{4^2 - 4(-1)(3)}}{2(-1)}$
Simplify.	$x = \frac{-4 \pm \sqrt{28}}{-2}$ $x = \frac{-4 \pm 2\sqrt{7}}{-2}$ $x = \frac{-2(2 \pm \sqrt{7})}{-2}$ $x = 2 \pm \sqrt{7}$
	The *x* intercepts are $(2 + \sqrt{7}, 0)$ and $(2 - \sqrt{7}, 0)$.

We will use the decimal approximations of the x-intercepts, so that we can locate these points on the graph.

$$\left(2 + \sqrt{7}, 0\right) \approx (4.6, 0) \qquad \left(2 - \sqrt{7}, 0\right) \approx (-0.6, 0)$$

Do these results agree with our graphs? See Figure 10.6.

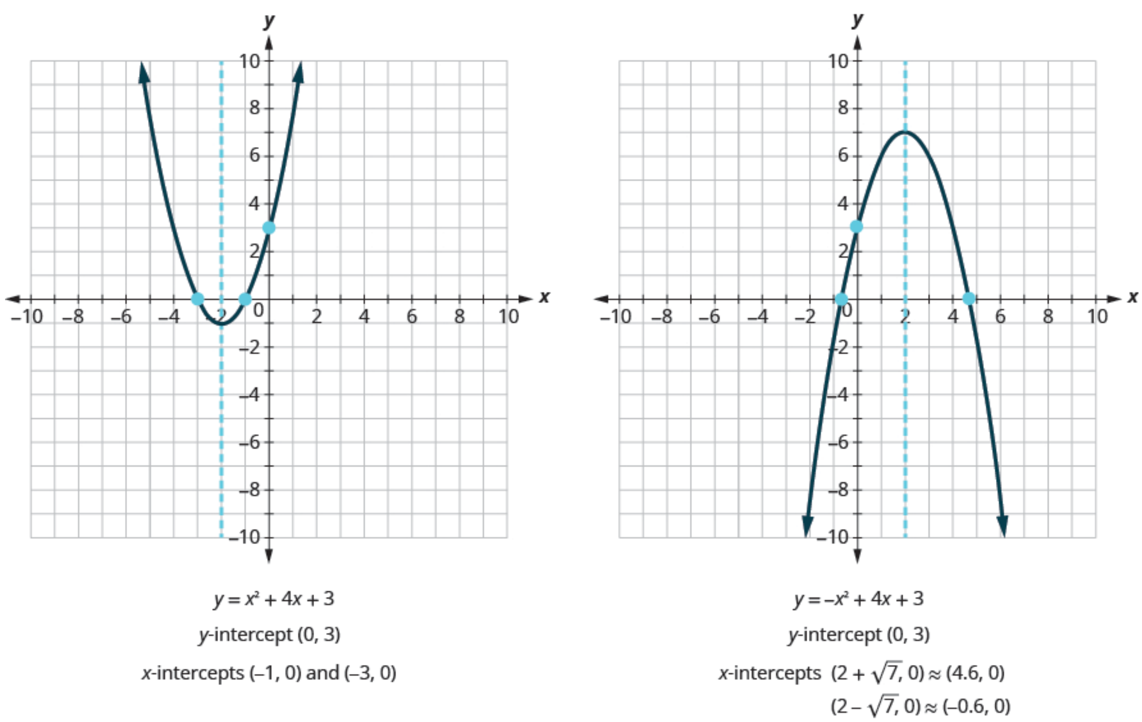


Figure 10.6

HOW TO : : FIND THE INTERCEPTS OF A PARABOLA.

To find the intercepts of a parabola with equation $y = ax^2 + bx + c$:

y-intercept	**x-intercepts**
Let $x = 0$ and solve for y.	Let $y = 0$ and solve for x.

EXAMPLE 10.46

Find the intercepts of the parabola $y = x^2 - 2x - 8$.

Solution

	$y = x^2 - 2x - 8$
To find the *y*-intercept, let $x = 0$ and solve for *y*.	$y = 0^2 - 2 \cdot 0 - 8$ $y = -8$
	When $x = 0$, then $y = -8$. The *y*-intercept is the point $(0, -8)$.
	$y = x^2 - 2x - 8$
To find the *x*-intercept, let $y = 0$ and solve for *x*.	$0 = x^2 - 2x - 8$
Solve by factoring.	$0 = (x - 4)(x + 2)$
	$0 = x - 4 \quad 0 = x + 2$ $4 = x \quad -2 = x$

When $y = 0$, then $x = 4$ or $x = -2$. The *x*-intercepts are the points $(4, 0)$ and $(-2, 0)$.

TRY IT : : 10.91 Find the intercepts of the parabola $y = x^2 + 2x - 8$.

TRY IT : : 10.92 Find the intercepts of the parabola $y = x^2 - 4x - 12$.

In this chapter, we have been solving quadratic equations of the form $ax^2 + bx + c = 0$. We solved for x and the results were the solutions to the equation.

We are now looking at quadratic equations in two variables of the form $y = ax^2 + bx + c$. The graphs of these equations are parabolas. The *x*-intercepts of the parabolas occur where $y = 0$.

For example:

Quadratic equation

$$\begin{aligned} x^2 - 2x - 15 &= 0 \\ (x-5)(x+3) &= 0 \end{aligned}$$

$$\begin{array}{rclcrcl} x - 5 &=& 0 & & x + 3 &=& 0 \\ x &=& 5 & & x &=& -3 \end{array}$$

Quadratic equation in two variables

$$y = x^2 - 2x - 15$$

$$\text{let } y = 0 \quad \begin{aligned} 0 &= x^2 - 2x - 15 \\ 0 &= (x-5)(x+3) \end{aligned}$$

$$\begin{array}{rclcrcl} x - 5 &=& 0 & & x + 3 &=& 0 \\ x &=& 5 & & x &=& -3 \end{array}$$

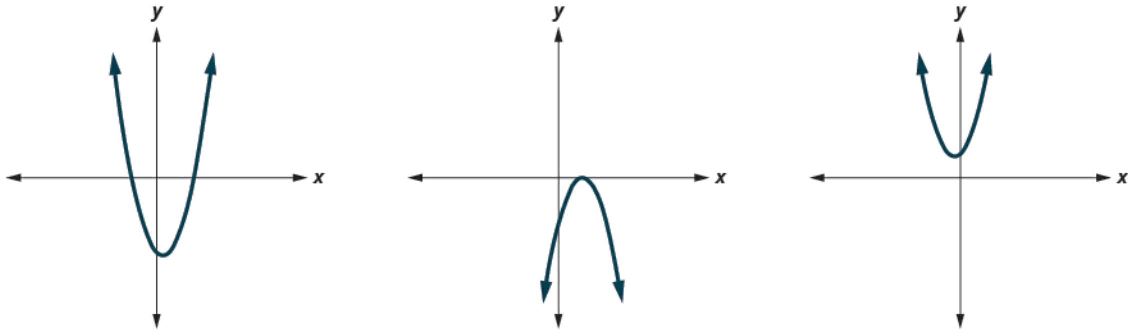
$$(5, 0) \text{ and } (-3, 0)$$

$$x\text{-intercepts}$$

The solutions of the quadratic equation are the x values of the *x*-intercepts.

Earlier, we saw that quadratic equations have 2, 1, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the equations give the *x*-intercepts of the graphs, the number of *x*-intercepts is the same as the number of solutions.

Previously, we used the discriminant to determine the number of solutions of a quadratic equation of the form $ax^2 + bx + c = 0$. Now, we can use the discriminant to tell us how many *x*-intercepts there are on the graph.



$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
Two solutions	One solution	No real solution
Two *x*-intercepts	One *x*-intercept	No *x*-intercept

Before you start solving the quadratic equation to find the values of the *x*-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

EXAMPLE 10.47

Find the intercepts of the parabola $y = 5x^2 + x + 4$.

Solution

	$y = 5x^2 + x + 4$
To find the *y*-intercept, let $x = 0$ and solve for *y*.	$y = 5 \cdot 0^2 + 0 + 4$ $y = 4$ When $x = 0$, then $y = 4$. The *y*-intercept is the point $(0, 4)$.
	$y = 5x^2 + x + 4$
To find the *x*-intercept, let $y = 0$ and solve for *x*.	$0 = 5x^2 + x + 4$
Find the value of the discriminant to predict the number of solutions and so *x*-intercepts.	$b^2 - 4ac$ $1^2 - 4 \cdot 5 \cdot 4$ $1 - 80$ -79
Since the value of the discriminant is negative, there is no real solution to the equation.	There are no *x*-intercepts.

TRY IT : : 10.93 Find the intercepts of the parabola $y = 3x^2 + 4x + 4$.

TRY IT : : 10.94 Find the intercepts of the parabola $y = x^2 - 4x - 5$.

EXAMPLE 10.48

Find the intercepts of the parabola $y = 4x^2 - 12x + 9$.

Solution

	$y = 4x^2 - 12x + 9$
To find the *y*-intercept, let $x = 0$ and solve for *y*.	$y = 4 \cdot 0^2 - 12 \cdot 0 + 9$ $y = 9$
	When $x = 0$, then $y = 9$. The *y*-intercept is the point $(0, 9)$.
	$y = 4x^2 - 12x + 9$
To find the *x*-intercept, let $y = 0$ and solve for *x*.	$0 = 4x^2 - 12x + 9$
Find the value of the discriminant to predict the number of solutions and so *x*-intercepts.	$b^2 - 4ac$ $2^2 - 4 \cdot 4 \cdot 9$ $144 - 144$ 0
	Since the value of the discriminant is 0, there is no real solution to the equation. So there is one *x*-intercept.
Solve the equation by factoring the perfect square trinomial.	$0 = (2x - 3)^2$
Use the Zero Product Property.	$0 = 2x - 3$
Solve for *x*.	$3 = 2x$ $\frac{3}{2} = x$
	When $y = 0$, then $\frac{3}{2} = x$.
	The *x*-intercept is the point $\left(\frac{3}{2}, 0\right)$.

TRY IT : : 10.95 Find the intercepts of the parabola $y = -x^2 - 12x - 36$.

TRY IT : : 10.96 Find the intercepts of the parabola $y = 9x^2 + 12x + 4$.

Graph Quadratic Equations in Two Variables

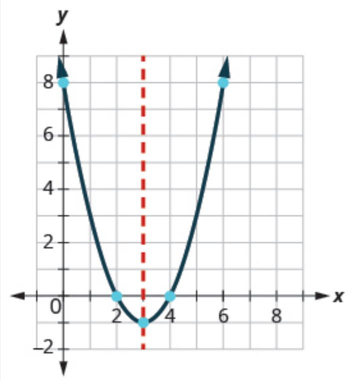
Now, we have all the pieces we need in order to graph a quadratic equation in two variables. We just need to put them together. In the next example, we will see how to do this.

EXAMPLE 10.49 HOW TO GRAPH A QUADRATIC EQUATION IN TWO VARIABLES

Graph $y = x^2 - 6x + 8$.

Solution

Step 1. Write the quadratic equation with y on one side.	This equation has y on one side.	$y = x^2 - 6x + 8$ $a = 1, b = -6, c = 8$
Step 2. Determine whether the parabola opens upward or downward.	Look at a in the equation. $y = x^2 - 6x + 8$ Since a is positive, the parabola opens upward.	 **The parabola opens upward.**
Step 3. Find the axis of symmetry.	$y = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2 \cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**
Step 4. Find the vertex.	The vertex is on the axis of symmetry. Substitute $x = 3$ into the equation and solve for y.	Vertex $y = x^2 - 6x + 8$ $y = (3)^2 - 6(3) + 8$ $y = -1$ **The vertex is (3, −1).**
Step 5. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We substitute $x = 0$ into the equation. We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	y-intercept $y = x^2 - 6x + 8$ $y = (0)^2 - 6(0) + 8$ $y = 8$ **The y-intercept is (0, 8).** Point symmetric to y-intercept **The point is (6, 8).**
Step 6. Find the x-intercepts.	We substitute $y = 0$ into the equation. We can solve this quadratic equation by factoring.	x-intercept $y = x^2 - 6x + 8$ $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2$ or $x = 4$ **The x-intercepts are (2, 0) and (4, 0).**

Step 7. Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the *y*-intercept. We connect these 5 points to sketch the parabola.	

TRY IT : : 10.97 Graph the parabola $y = x^2 + 2x - 8$.

TRY IT : : 10.98 Graph the parabola $y = x^2 - 8x + 12$.

HOW TO : : GRAPH A QUADRATIC EQUATION IN TWO VARIABLES.

Step 1. Write the quadratic equation with y on one side.

Step 2. Determine whether the parabola opens upward or downward.

Step 3. Find the axis of symmetry.

Step 4. Find the vertex.

Step 5. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.

Step 6. Find the *x*-intercepts.

Step 7. Graph the parabola.

We were able to find the *x*-intercepts in the last example by factoring. We find the *x*-intercepts in the next example by factoring, too.

EXAMPLE 10.50

Graph $y = -x^2 + 6x - 9$.

Solution

The equation *y* has on one side.	$y = ax^2 + bx + c$ $y = -x^2 + 6x - 9$

Since *a* is -1, the parabola opens downward.

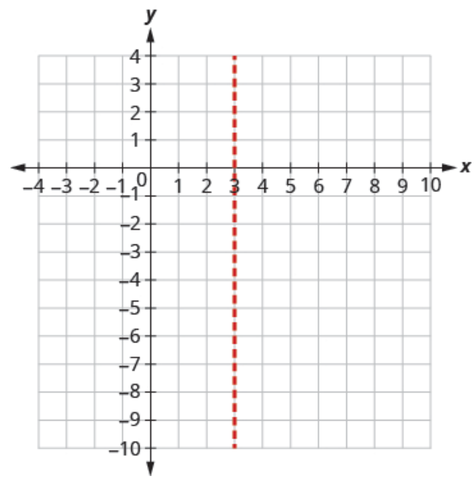
To find the axis of symmetry, find $x = -\frac{b}{2a}$.

$x = -\frac{b}{2a}$

$x = -\frac{6}{2(-1)}$

$x = 3$

The axis of symmetry is $x = 3$. The vertex is on the line $x = 3$.



Find *y* when $x = 3$.

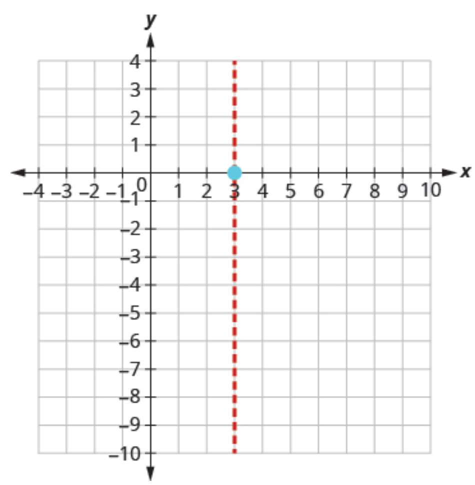
$y = -x^2 + 6x - 9$

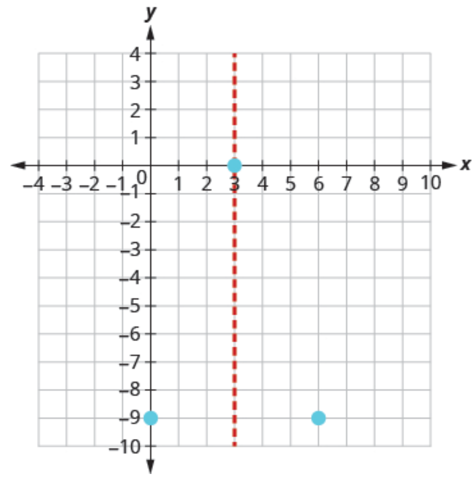
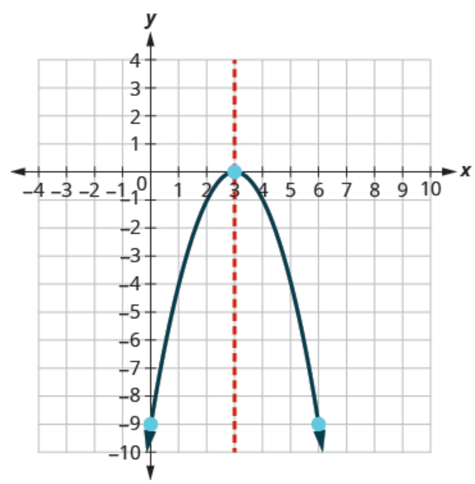
$y = -3^2 + 6 \cdot 3 - 9$

$y = -9 + 18 - 9$

$y = 0$

The vertex is $(3, 0)$.



The y-intercept occurs when $x = 0$.	$y = -x^2 + 6x - 9$
Substitute $x = 0$.	$y = -0^2 + 6 \cdot 0 - 9$
Simplify.	$y = -9$
	The y-intercept is $(0, -9)$.
The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$. Point symmetric to the y-intercept is $(6, -9)$	
The x-intercept occurs when $y = 0$.	$y = -x^2 + 6x - 9$
Substitute $y = 0$.	$0 = -x^2 + 6x - 9$
Factor the GCF.	$0 = -(x^2 - 6x + 9)$
Factor the trinomial.	$0 = -(x - 3)^2$
Solve for x.	$x = 3$
Connect the points to graph the parabola.	

TRY IT : : 10.99 Graph the parabola $y = -3x^2 + 12x - 12$.

TRY IT : : 10.100 Graph the parabola $y = 25x^2 + 10x + 1$.

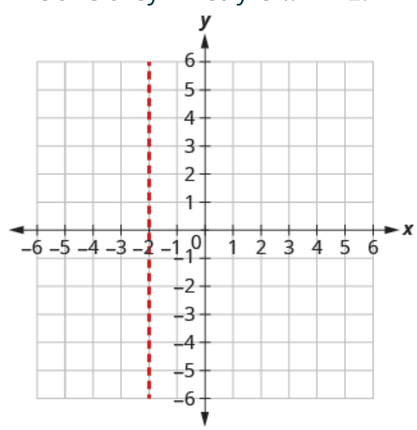
For the graph of $y = -x^2 + 6x - 9$, the vertex and the x-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0 = -x^2 + 6x - 9$ is 0, so there is only one solution. That means there is only one *x*-intercept, and it is the vertex of the parabola.

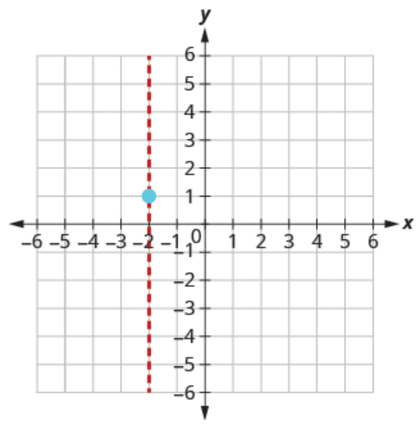
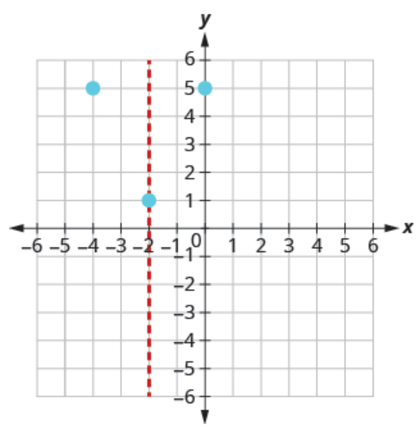
How many *x*-intercepts would you expect to see on the graph of $y = x^2 + 4x + 5$?

EXAMPLE 10.51

Graph $y = x^2 + 4x + 5$.

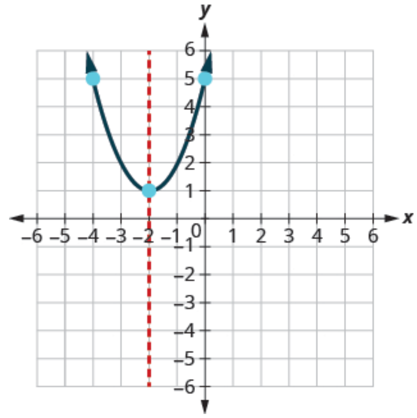
Solution

The equation has *y* on one side.	$\overset{y\,=\,ax^2\,+\,bx\,+\,c}{y = x^2 + 4x + 5}$
Since a is 1, the parabola opens upward.	
To find the axis of symmetry, find $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$ $x = -\frac{4}{2(1)}$ $x = -2$ The axis of symmetry is $x = -2$. 
The vertex is on the line $x = -2$.	

Find *y* when $x = -2$.	$y = x^2 + 4x + 5$ $y = (-2)^2 + 4 \cdot (-2) + 5$ $y = 4 - 8 + 5$ $y = 1$ The vertex is $(-2, 1)$. 
The *y*-intercept occurs when $x = 0$. Substitute $x = 0$. Simplify. The point $(0, 5)$ is two units to the right of the line of symmetry. The point two units to the left of the line of symmetry is $(-4, 5)$.	$y = x^2 + 4x + 5$ $y = (0)^2 + 4(0) + 5$ $y = 5$ The *y*-intercept is $(0, 5)$.  Point symmetric to the *y*- intercept is $(-4, 5)$.
The *x*- intercept occurs when $y = 0$.	
Substitute $y = 0$. Test the discriminant.	$y = x^2 + 4x + 5$ $0 = x^2 + 4x + 5$

$$b^2 - 4ac$$
$$4^2 - 4 \cdot 15$$
$$16 - 20$$
$$-4$$

Since the value of the discriminant is negative, there is no solution and so no *x*- intercept.
Connect the points to graph the parabola. You may want to choose two more points for greater accuracy.



> **TRY IT : : 10.101** Graph the parabola $y = 2x^2 - 6x + 5$.

> **TRY IT : : 10.102** Graph the parabola $y = -2x^2 - 1$.

Finding the *y*-intercept by substituting $x = 0$ into the equation is easy, isn't it? But we needed to use the Quadratic Formula to find the *x*-intercepts in **Example 10.51**. We will use the Quadratic Formula again in the next example.

EXAMPLE 10.52

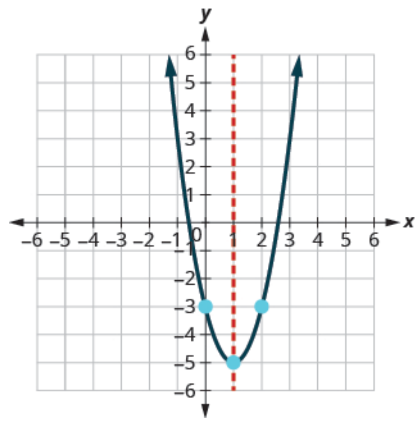
Graph $y = 2x^2 - 4x - 3$.

Solution

	$y = ax^2 + bx + c$ $y = 2x^2 - 4x - 3$
The equation y has one side. Since a is 2, the parabola opens upward.	
To find the axis of symmetry, find $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$ $x = -\frac{-4}{2 \cdot 2}$ $x = 1$ The axis of symmetry is $x = 1$.
The vertex on the line $x = 1$.	$y = 2x^2 - 4x - 3$

Find y when $x = 1$.	$y = 2(1)^2 - 4 \cdot (1) - 3$ $y = 2 - 4 - 3$ $y = -5$ The vertex is $(1, -5)$.
The y-intercept occurs when $x = 0$.	$y = 2x^2 - 4x - 3$
Substitute $x = 0$.	$y = 2 \cdot 0^2 - 4 \cdot 0 - 3$
Simplify.	$y = -3$ The y-intercept is $(0, -3)$.
The point $(0, -3)$ is one unit to the left of the line of symmetry. The point one unit to the right of the line of symmetry is $(2, -3)$	Point symmetric to the y-intercept is $(2, -3)$.
The x-intercept occurs when $y = 0$.	$y = 2x^2 - 4x - 3$
Substitute $y = 0$.	$0 = 2x^2 - 4x - 3$
Use the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute in the values of *a, b, c.*	$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot (-3)}}{2 \cdot 2}$
Simplify.	$x = \frac{4 \pm \sqrt{16 + 24}}{4}$
Simplify inside the radical.	$x = \frac{4 \pm \sqrt{40}}{4}$
Simplify the radical.	$x = \frac{4 \pm 2\sqrt{10}}{4}$
Factor the GCF.	$x = \frac{2(2 \pm \sqrt{10})}{4}$
Remove common factors.	$x = \frac{2 \pm \sqrt{10}}{2}$
Write as two equations.	$x = \frac{2 + \sqrt{10}}{2}, x = \frac{2 - \sqrt{10}}{2}$
Approximate the values.	$x \approx 2.5, x \approx -0.6$
	The approximate values of the x-intercepts are $(2.5, 0)$ and $(-0.6, 0)$.

Graph the parabola using the points found.



> **TRY IT : :** 10.103 Graph the parabola $y = 5x^2 + 10x + 3$.

> **TRY IT : :** 10.104 Graph the parabola $y = -3x^2 - 6x + 5$.

Solve Maximum and Minimum Applications

Knowing that the vertex of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic equation. The *y*-coordinate of the vertex is the minimum *y*-value of a parabola that opens upward. It is the maximum *y*-value of a parabola that opens downward. See **Figure 10.7**.

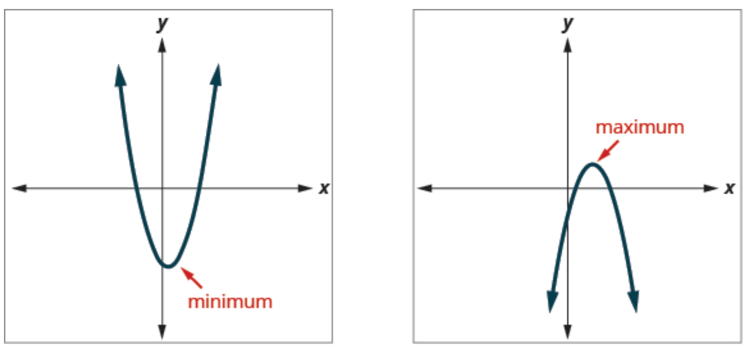


Figure 10.7

Minimum or Maximum Values of a Quadratic Equation

The ***y*-coordinate of the vertex** of the graph of a quadratic equation is the

- minimum value of the quadratic equation if the parabola opens upward.
- maximum value of the quadratic equation if the parabola opens downward.

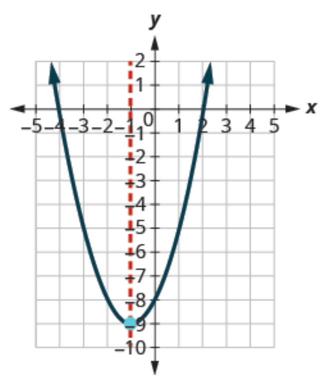
EXAMPLE 10.53

Find the minimum value of the quadratic equation $y = x^2 + 2x - 8$.

Solution

$$y = x^2 + 2x - 8$$

Since *a* is positive, the parabola opens upward.

The quadratic equation has a minimum.	
Find the axis of symmetry.	$x = -\frac{b}{2a}$ $x = -\frac{2}{2 \cdot 1}$ $x = -1$ The axis of symmetry is $x = -1$.
The vertex is on the line $x = -1$.	$y = x^2 + 2x - 8$
Find *y* when $x = -1$.	$y = (-1)^2 + 2 \cdot (-1) - 8$ $y = 1 - 2 - 8$ $y = -9$ The vertex is $(-1, -9)$.
Since the parabola has a minimum, the *y*-coordinate of the vertex is the minimum *y*-value of the quadratic equation.	
The minimum value of the quadratic is -9 and it occurs when $x = -1$.	
Show the graph to verify the result.	

TRY IT : : 10.105 Find the maximum or minimum value of the quadratic equation $y = x^2 - 8x + 12$.

TRY IT : : 10.106 Find the maximum or minimum value of the quadratic equation $y = -4x^2 + 16x - 11$.

We have used the formula

$$h = -16t^2 + v_0 t + h_0$$

to calculate the height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds.

This formula is a quadratic equation in the variable t, so its graph is a parabola. By solving for the coordinates of the vertex, we can find how long it will take the object to reach its maximum height. Then, we can calculate the maximum height.

EXAMPLE 10.54

The quadratic equation $h = -16t^2 + v_0 t + h_0$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

ⓐ How many seconds will it take the volleyball to reach its maximum height?

ⓑ Find the maximum height of the volleyball.

Solution

$h = -16t^2 + 176t + 4$

Since a is negative, the parabola opens downward.
The quadratic equation has a maximum.

ⓐ

Find the axis of symmetry.	$t = -\frac{b}{2a}$ $t = -\frac{176}{2(-16)}$ $t = 5.5$ The axis of symmetry is $t = 5.5$.
The vertex is on the line $t = 5.5$.	The maximum occurs when $t = 5.5$ seconds.

ⓑ

Find *h* when $t = 5.5$.	$h = -16t^2 + 176t + 4$ $h = -16(5.5)^2 + 176 \cdot (5.5) + 4$
Use a calculator to simplify.	$h = 488$
	The vertex is $(5.5, 488)$.
Since the parabola has a maximum, the *h*-coordinate of the vertex is the maximum *y*-value of the quadratic equation.	The maximum value of the quadratic is 488 feet and it occurs when $t = 5.5$ seconds.

TRY IT : : 10.107

The quadratic equation $h = -16t^2 + 128t + 32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height? Round answers to the nearest tenth.

TRY IT : : 10.108

A toy rocket shot upward from the ground at a rate of 208 ft/sec has the quadratic equation of $h = -16t^2 + 208t$. When will the rocket reach its maximum height? What will be the maximum height? Round answers to the nearest tenth.

MEDIA : :

Access these online resources for additional instruction and practice graphing quadratic equations:

- **Graphing Quadratic Functions (https://openstax.org/l/25Graphquad1)**
- **How do you graph a quadratic function? (https://openstax.org/l/25Graphquad2)**
- **Graphing Quadratic Equations (https://openstax.org/l/25Graphquad3)**

10.5 EXERCISES

Practice Makes Perfect

Recognize the Graph of a Quadratic Equation in Two Variables

In the following exercises, graph:

163. $y = x^2 + 3$

164. $y = -x^2 + 1$

In the following exercises, determine if the parabola opens up or down.

165. $y = -2x^2 - 6x - 7$

166. $y = 6x^2 + 2x + 3$

167. $y = 4x^2 + x - 4$

168. $y = -9x^2 - 24x - 16$

Find the Axis of Symmetry and Vertex of a Parabola

In the following exercises, find ⓐ the axis of symmetry and ⓑ the vertex.

169. $y = x^2 + 8x - 1$

170. $y = x^2 + 10x + 25$

171. $y = -x^2 + 2x + 5$

172. $y = -2x^2 - 8x - 3$

Find the Intercepts of a Parabola

In the following exercises, find the x- and y-intercepts.

173. $y = x^2 + 7x + 6$

174. $y = x^2 + 10x - 11$

175. $y = -x^2 + 8x - 19$

176. $y = x^2 + 6x + 13$

177. $y = 4x^2 - 20x + 25$

178. $y = -x^2 - 14x - 49$

Graph Quadratic Equations in Two Variables

In the following exercises, graph by using intercepts, the vertex, and the axis of symmetry.

179. $y = x^2 + 6x + 5$

180. $y = x^2 + 4x - 12$

181. $y = x^2 + 4x + 3$

182. $y = x^2 - 6x + 8$

183. $y = 9x^2 + 12x + 4$

184. $y = -x^2 + 8x - 16$

185. $y = -x^2 + 2x - 7$

186. $y = 5x^2 + 2$

187. $y = 2x^2 - 4x + 1$

188. $y = 3x^2 - 6x - 1$

189. $y = 2x^2 - 4x + 2$

190. $y = -4x^2 - 6x - 2$

191. $y = -x^2 - 4x + 2$

192. $y = x^2 + 6x + 8$

193. $y = 5x^2 - 10x + 8$

194. $y = -16x^2 + 24x - 9$

195. $y = 3x^2 + 18x + 20$

196. $y = -2x^2 + 8x - 10$

Solve Maximum and Minimum Applications

In the following exercises, find the maximum or minimum value.

197. $y = 2x^2 + x - 1$

198. $y = -4x^2 + 12x - 5$

199. $y = x^2 - 6x + 15$

200. $y = -x^2 + 4x - 5$

201. $y = -9x^2 + 16$

202. $y = 4x^2 - 49$

In the following exercises, solve. Round answers to the nearest tenth.

203. An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic equation $h = -16t^2 + 168t + 45$ to find how long it will take the arrow to reach its maximum height, and then find the maximum height.

204. A stone is thrown vertically upward from a platform that is 20 feet high at a rate of 160 ft/sec. Use the quadratic equation $h = -16t^2 + 160t + 20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.

205. A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40 - x$ computers each week. The quadratic equation $R = -x^2 + 40x$ is used to find the revenue, R, received when the selling price of a computer is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

206. A retailer who sells backpacks estimates that, by selling them for x dollars each, he will be able to sell $100 - x$ backpacks a month. The quadratic equation $R = -x^2 + 100x$ is used to find the R received when the selling price of a backpack is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

207. A rancher is going to fence three sides of a corral next to a river. He needs to maximize the corral area using 240 feet of fencing. The quadratic equation $A = x(240 - 2x)$ gives the area of the corral, A, for the length, x, of the corral along the river. Find the length of the corral along the river that will give the maximum area, and then find the maximum area of the corral.

208. A veterinarian is enclosing a rectangular outdoor running area against his building for the dogs he cares for. He needs to maximize the area using 100 feet of fencing. The quadratic equation $A = x(100 - 2x)$ gives the area, A, of the dog run for the length, x, of the building that will border the dog run. Find the length of the building that should border the dog run to give the maximum area, and then find the maximum area of the dog run.

Everyday Math

209. In the previous set of exercises, you worked with the quadratic equation $R = -x^2 + 40x$ that modeled the revenue received from selling computers at a price of x dollars. You found the selling price that would give the maximum revenue and calculated the maximum revenue. Now you will look at more characteristics of this model.

ⓐ Graph the equation $R = -x^2 + 40x$. ⓑ Find the values of the *x*-intercepts.

210. In the previous set of exercises, you worked with the quadratic equation $R = -x^2 + 100x$ that modeled the revenue received from selling backpacks at a price of x dollars. You found the selling price that would give the maximum revenue and calculated the maximum revenue. Now you will look at more characteristics of this model.

ⓐ Graph the equation $R = -x^2 + 100x$. ⓑ Find the values of the *x*-intercepts.

Writing Exercises

211. For the revenue model in **Exercise 10.205** and **Exercise 10.209**, explain what the *x*-intercepts mean to the computer store owner.

212. For the revenue model in **Exercise 10.206** and **Exercise 10.210**, explain what the *x*-intercepts mean to the backpack retailer.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
recognize the graph of a quadratic equation in two variables.			
find the axis of symmetry and vertex of a parabola.			
find the intercepts of a parabola.			
graph quadratic equations in two variables.			
solve maximum and minimum applications.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

CHAPTER 10 REVIEW

KEY TERMS

axis of symmetry The axis of symmetry is the vertical line passing through the middle of the parabola $y = ax^2 + bx + c.$

completing the square Completing the square is a method used to solve quadratic equations.

consecutive even integers Consecutive even integers are even integers that follow right after one another. If an even integer is represented by n, the next consecutive even integer is $n + 2$, and the next after that is $n + 4$.

consecutive odd integers Consecutive odd integers are odd integers that follow right after one another. If an odd integer is represented by n, the next consecutive odd integer is $n + 2$, and the next after that is $n + 4$.

discriminant In the Quadratic Formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ the quantity $b^2 - 4ac$ is called the discriminant.

parabola The graph of a quadratic equation in two variables is a parabola.

quadratic equation A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0.$

quadratic equation in two variables A quadratic equation in two variables, where $a, b,$ and c are real numbers and $a \neq 0$ is an equation of the form $y = ax^2 + bx + c.$

Square Root Property The Square Root Property states that, if $x^2 = k$ and $k \geq 0$, then $x = \sqrt{k} \text{ or } x = -\sqrt{k}.$

vertex The point on the parabola that is on the axis of symmetry is called the *vertex* of the parabola; it is the lowest or highest point on the parabola, depending on whether the parabola opens upwards or downwards.

***x*-intercepts of a parabola** The *x*-intercepts are the points on the parabola where $y = 0.$

***y*-intercept of a parabola** The *y*-intercept is the point on the parabola where $x = 0.$

KEY CONCEPTS

10.1 Solve Quadratic Equations Using the Square Root Property

- Square Root Property
  If $x^2 = k$, and $k \geq 0$, then $x = \sqrt{k} \text{ or } x = -\sqrt{k}$.

10.2 Solve Quadratic Equations by Completing the Square

- Binomial Squares Pattern If a, b are real numbers,

  $$(a + b)^2 = a^2 + 2ab + b^2$$

  $$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2 \times (\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

  $$(a - b)^2 = a^2 - 2ab + b^2$$

  $$\underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2 \times (\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

- Complete a Square
  To complete the square of $x^2 + bx$:

  Step 1. Identify b, the coefficient of x.

  Step 2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.

Step 3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$.

10.3 Solve Quadratic Equations Using the Quadratic Formula

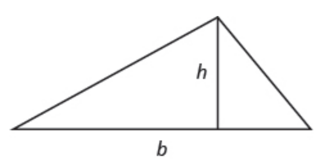
- **Quadratic Formula** The solutions to a quadratic equation of the form $ax^2 + bx + c = 0, \quad a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

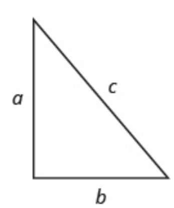
- **Solve a Quadratic Equation Using the Quadratic Formula**
To solve a quadratic equation using the Quadratic Formula.
Step 1. Write the quadratic formula in standard form. Identify the a, b, c values.
Step 2. Write the quadratic formula. Then substitute in the values of a, b, c.
Step 3. Simplify.
Step 4. Check the solutions.
- **Using the Discriminant, $b^2 - 4ac$, to Determine the Number of Solutions of a Quadratic Equation**
For a quadratic equation of the form $ax^2 + bx + c = 0, \quad a \neq 0,$
    - if $b^2 - 4ac > 0$, the equation has 2 solutions.
    - if $b^2 - 4ac = 0$, the equation has 1 solution.
    - if $b^2 - 4ac < 0$, the equation has no real solutions.
- **To identify the most appropriate method to solve a quadratic equation:**
Step 1. Try Factoring first. If the quadratic factors easily this method is very quick.
Step 2. Try the Square Root Property next. If the equation fits the form $ax^2 = k$ or $a(x - h)^2 = k$, it can easily be solved by using the Square Root Property.
Step 3. Use the Quadratic Formula. Any other quadratic equation is best solved by using the Quadratic Formula.

10.4 Solve Applications Modeled by Quadratic Equations

- **Area of a Triangle** For a triangle with base, b, and height, h, the area, A, is given by the formula: $A = \frac{1}{2}bh$



- **Pythagorean Theorem** In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypothenuse, $a^2 + b^2 = c^2$



- **Projectile motion** The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds can be modeled by the formula:

$$h = -16t^2 + v_0 t$$

10.5 Graphing Quadratic Equations

- **The graph of every quadratic equation is a parabola.**
- **Parabola Orientation** For the quadratic equation $y = ax^2 + bx + c$, if
  - $a > 0$, the parabola opens upward.
  - $a < 0$, the parabola opens downward.
- **Axis of Symmetry and Vertex of a Parabola** For a parabola with equation $y = ax^2 + bx + c$:
  - The axis of symmetry of a parabola is the line $x = -\frac{b}{2a}$.
  - The vertex is on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.
  - To find the *y*-coordinate of the vertex we substitute $x = -\frac{b}{2a}$ into the quadratic equation.
- **Find the Intercepts of a Parabola** To find the intercepts of a parabola with equation $y = ax^2 + bx + c$:

***y*-intercept**	***x*-intercepts**
Let $x = 0$ and solve for y.	Let $y = 0$ and solve for x.

- **To Graph a Quadratic Equation in Two Variables**
  Step 1. Write the quadratic equation with y on one side.
  Step 2. Determine whether the parabola opens upward or downward.
  Step 3. Find the axis of symmetry.
  Step 4. Find the vertex.
  Step 5. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
  Step 6. Find the *x*-intercepts.
  Step 7. Graph the parabola.
- **Minimum or Maximum Values of a Quadratic Equation**
  - The *y*-**coordinate of the vertex** of the graph of a quadratic equation is the
  - **minimum** value of the quadratic equation if the parabola opens upward.
  - **maximum** value of the quadratic equation if the parabola opens downward.

REVIEW EXERCISES

10.1 10.1 Solve Quadratic Equations Using the Square Root Property

In the following exercises, solve using the Square Root Property.

213. $x^2 = 100$

214. $y^2 = 144$

215. $m^2 - 40 = 0$

216. $n^2 - 80 = 0$

217. $4a^2 = 100$

218. $2b^2 = 72$

219. $r^2 + 32 = 0$

220. $t^2 + 18 = 0$

221. $\frac{4}{3}v^2 + 4 = 28$

222. $\frac{2}{3}w^2 - 20 = 30$

223. $5c^2 + 3 = 19$

224. $3d^2 - 6 = 43$

In the following exercises, solve using the Square Root Property.

225. $(p - 5)^2 + 3 = 19$

226. $(q + 4)^2 = 9$

227. $(u + 1)^2 = 45$

228. $(z-5)^2 = 50$

229. $\left(x-\frac{1}{4}\right)^2 = \frac{3}{16}$

230. $\left(y-\frac{2}{3}\right)^2 = \frac{2}{9}$

231. $(m-7)^2 + 6 = 30$

232. $(n-4)^2 - 50 = 150$

233. $(5c+3)^2 = -20$

234. $(4c-1)^2 = -18$

235. $m^2 - 6m + 9 = 48$

236. $n^2 + 10n + 25 = 12$

237. $64a^2 + 48a + 9 = 81$

238. $4b^2 - 28b + 49 = 25$

10.2 10.2 Solve Quadratic Equations Using Completing the Square

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

239. $x^2 + 22x$

240. $y^2 + 6y$

241. $m^2 - 8m$

242. $n^2 - 10n$

243. $a^2 - 3a$

244. $b^2 + 13b$

245. $p^2 + \frac{4}{5}p$

246. $q^2 - \frac{1}{3}q$

In the following exercises, solve by completing the square.

247. $c^2 + 20c = 21$

248. $d^2 + 14d = -13$

249. $x^2 - 4x = 32$

250. $y^2 - 16y = 36$

251. $r^2 + 6r = -100$

252. $t^2 - 12t = -40$

253. $v^2 - 14v = -31$

254. $w^2 - 20w = 100$

255. $m^2 + 10m - 4 = -13$

256. $n^2 - 6n + 11 = 34$

257. $a^2 = 3a + 8$

258. $b^2 = 11b - 5$

259. $(u+8)(u+4) = 14$

260. $(z-10)(z+2) = 28$

261. $3p^2 - 18p + 15 = 15$

262. $5q^2 + 70q + 20 = 0$

263. $4y^2 - 6y = 4$

264. $2x^2 + 2x = 4$

265. $3c^2 + 2c = 9$

266. $4d^2 - 2d = 8$

10.3 10.3 Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

267. $4x^2 - 5x + 1 = 0$

268. $7y^2 + 4y - 3 = 0$

269. $r^2 - r - 42 = 0$

270. $t^2 + 13t + 22 = 0$

271. $4v^2 + v - 5 = 0$

272. $2w^2 + 9w + 2 = 0$

273. $3m^2 + 8m + 2 = 0$

274. $5n^2 + 2n - 1 = 0$

275. $6a^2 - 5a + 2 = 0$

276. $4b^2 - b + 8 = 0$

277. $u(u-10) + 3 = 0$

278. $5z(z-2) = 3$

279. $\frac{1}{8}p^2 - \frac{1}{5}p = -\frac{1}{20}$

280. $\frac{2}{5}q^2 + \frac{3}{10}q = \frac{1}{10}$

281. $4c^2 + 4c + 1 = 0$

282. $9d^2 - 12d = -4$

In the following exercises, determine the number of solutions to each quadratic equation.

283.

ⓐ $9x^2 - 6x + 1 = 0$
ⓑ $3y^2 - 8y + 1 = 0$
ⓒ $7m^2 + 12m + 4 = 0$
ⓓ $5n^2 - n + 1 = 0$

284.

ⓐ $5x^2 - 7x - 8 = 0$
ⓑ $7x^2 - 10x + 5 = 0$
ⓒ $25x^2 - 90x + 81 = 0$
ⓓ $15x^2 - 8x + 4 = 0$

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation.

285.

ⓐ $16r^2 - 8r + 1 = 0$
ⓑ $5t^2 - 8t + 3 = 9$
ⓒ $3(c + 2)^2 = 15$

286.

ⓐ $4d^2 + 10d - 5 = 21$
ⓑ $25x^2 - 60x + 36 = 0$
ⓒ $6(5v - 7)^2 = 150$

10.4 10.4 Solve Applications Modeled by Quadratic Equations

In the following exercises, solve by using methods of factoring, the square root principle, or the quadratic formula.

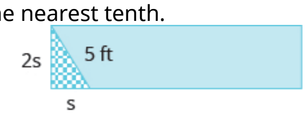
287. Find two consecutive odd numbers whose product is 323.

288. Find two consecutive even numbers whose product is 624.

289. A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.

290. Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.

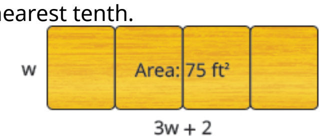
291. A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.



292. A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

293. The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

294. For Sophia's graduation party, several tables of the same width will be arranged end to end to give a serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth. Round answer to the nearest tenth.



295. A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

296. A bullet is fired straight up from the ground at a velocity of 320 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the bullet will reach 800 feet. Round to the nearest tenth.

10.5 10.5 Graphing Quadratic Equations in Two Variables

In the following exercises, graph by plotting point.

297. Graph $y = x^2 - 2$

298. Graph $y = -x^2 + 3$

In the following exercises, determine if the following parabolas open up or down.

299. $y = -3x^2 + 3x - 1$

300. $y = 5x^2 + 6x + 3$

301. $y = x^2 + 8x - 1$

302. $y = -4x^2 - 7x + 1$

In the following exercises, find ⓐ the axis of symmetry and ⓑ the vertex.

303. $y = -x^2 + 6x + 8$

304. $y = 2x^2 - 8x + 1$

In the following exercises, find the x- and y-intercepts.

305. $y = x^2 - 4x + 5$

306. $y = x^2 - 8x + 15$

307. $y = x^2 - 4x + 10$

308. $y = -5x^2 - 30x - 46$

309. $y = 16x^2 - 8x + 1$

310. $y = x^2 + 16x + 64$

In the following exercises, graph by using intercepts, the vertex, and the axis of symmetry.

311. $y = x^2 + 8x + 15$

312. $y = x^2 - 2x - 3$

313. $y = -x^2 + 8x - 16$

314. $y = 4x^2 - 4x + 1$

315. $y = x^2 + 6x + 13$

316. $y = -2x^2 - 8x - 12$

317. $y = -4x^2 + 16x - 11$

318. $y = x^2 + 8x + 10$

In the following exercises, find the minimum or maximum value.

319. $y = 7x^2 + 14x + 6$

320. $y = -3x^2 + 12x - 10$

In the following exercises, solve. Rounding answers to the nearest tenth.

321. A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.

322. A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, A, of the yard for the length, x, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.

PRACTICE TEST

323. Use the Square Root Property to solve the quadratic equation: $3(w+5)^2 = 27$.

324. Use Completing the Square to solve the quadratic equation: $a^2 - 8a + 7 = 23$.

325. Use the Quadratic Formula to solve the quadratic equation: $2m^2 - 5m + 3 = 0$.

Solve the following quadratic equations. Use any method.

326. $8v^2 + 3 = 35$

327. $3n^2 + 8n + 3 = 0$

328. $2b^2 + 6b - 8 = 0$

329. $x(x+3) + 12 = 0$

330. $\frac{4}{3}y^2 - 4y + 3 = 0$

Use the discriminant to determine the number of solutions of each quadratic equation.

331. $6p^2 - 13p + 7 = 0$

332. $3q^2 - 10q + 12 = 0$

Solve by factoring, the Square Root Property, or the Quadratic Formula.

333. Find two consecutive even numbers whose product is 360.

334. The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

For each parabola, find ⓐ which ways it opens, ⓑ the axis of symmetry, ⓒ the vertex, ⓓ the x- and y-intercepts, and ⓔ the maximum or minimum value.

335. $y = 3x^2 + 6x + 8$

336. $y = x^2 - 4$

337. $y = x^2 + 10x + 24$

338. $y = -3x^2 + 12x - 8$

339. $y = -x^2 - 8x + 16$

Graph the following parabolas by using intercepts, the vertex, and the axis of symmetry.

340. $y = 2x^2 + 6x + 2$

341. $y = 16x^2 + 24x + 9$

Solve.

342. A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h = -16t^2 + 86t$, find how long it will take the balloon to reach the maximum height and then find the maximum height. Round to the nearest tenth.